universität innsbruck

Automata and Logic	25W	LVA 703026 + 703027
Lecture 13		January 23, 2026

Solved exercises must be marked and solutions (as a single PDF file) uploaded in OLAT. The (strict) deadline is 7 am on January 23.

Exercises

⟨2⟩ 1. For $n > 0$ let $L_n = \{x \in \{a\}^* \mid |x| = 1 \mod n\}$.

   (a) Show that for all $n > 0$ there exists an NFA N with n states accepting L_n.

   (b) Show that for all $n > 0$ no NFA with less than n states accepts L_n.

⟨3⟩ 2. Let $M = (Q, \Sigma, \Delta, s, F)$ be an AFA. Consider the following definition of an extended transition function $\widehat{\Delta} \colon \mathbb{B}^+(Q) \times \Sigma^* \to \mathbb{B}^+(Q)$:

$$\begin{aligned} \widehat{\Delta}(q, \epsilon) &= q & \widehat{\Delta}(\varphi \vee \psi, x) &= \widehat{\Delta}(\varphi, x) \vee \widehat{\Delta}(\psi, x) \\ \widehat{\Delta}(q, ax) &= \widehat{\Delta}(\Delta(q, a), x) & \widehat{\Delta}(\varphi \wedge \psi, x) &= \widehat{\Delta}(\varphi, x) \wedge \widehat{\Delta}(\psi, x) \\ \widehat{\Delta}(\top, x) &= \top & \widehat{\Delta}(\bot, x) &= \bot \end{aligned}$$

   (a) Consider a string $x \in \Sigma^*$, a formula $\varphi \in \mathbb{B}^+$ as well as a set $P \subseteq Q$. Show that $P \vDash \widehat{\Delta}(\varphi, x)$ if and only if there exists a minimal $R \vDash \varphi$ as well as runs r_q in M on x starting at each $q \in R$ such that $\bigcup\{\text{leaf}(r_q) \mid q \in R\} \subseteq P$ where $\text{leaf}(r_q)$ denotes the set of leaves of r_q at level $n = |x|$.

   (b) Using the result from (a), show that $L(M) = \{x \mid F \vDash \widehat{\Delta}(s, x)\}$.

⟨3⟩ 3. Let $M = (Q, \Sigma, \Delta, s, F)$ be an AFA and $\overline{M} = (Q, \Sigma, \overline{\Delta}, s, Q - F)$ its dual. In this exercise we prove $L(\overline{M}) = {\sim} L(M)$, i.e., the theorem on slide 20.

   (a) Show that for all $R \subseteq Q$ and $\varphi \in \mathbb{B}^+$, $R \nvDash \varphi$ if and only if $Q \setminus R \vDash \overline{\varphi}$.

   (b) Show that for all $\varphi \in \mathbb{B}^+$ and $x \in \Sigma^*$,

$$\widehat{\overline{\Delta}}(\varphi, x) = \overline{\widehat{\Delta}(\varphi, x)}$$

   where $\widehat{\Delta}$ is defined as in Exercise 3. *Hint:* Use nested induction with induction on $|x|$ on the outer level as well as induction on φ on the inner level. Try to find out how $\widehat{\overline{\Delta}}$ has to be defined while conducting the proof!

   (c) Using the result from 3(b) as well as (a) and (b), show that $L(\overline{M}) = {\sim} L(M)$. Note that you can use 3(b) also for $\widehat{\overline{\Delta}}$ as 3(a) can be proven in exactly the same way when $P \vDash \widehat{\Delta}(\varphi, x)$ and $R \vDash \varphi$ are replaced by $P \vDash \widehat{\overline{\Delta}}(\varphi, x)$ and $R \vDash \overline{\varphi}$, respectively.

⟨2⟩ 4. Consider the LTL formula $\varphi = \neg(\mathrm{G}\, a \vee ((\neg\, \mathrm{X}\, a) \,\mathrm{U}\, \neg b))$.

   (a) Transform ϕ into negation normal form ψ.

   (b) Use the construction from the lecture to compute the alternating Büchi automaton A_ψ.

   (c) Which of the following traces are accepted by A_ψ?

   i. $\{b\}\{a\}\varnothing^\omega$ ii. $\{a, b\}\{a\}^\omega$ iii. $\{a, b\}\{b\}^\omega$