universität innsbruck

Automata and Logic

Aart Middeldorp and Samuel Frontull

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  2. Constraint Solving
  3. Cryptography
  4. High−Performance Computing
  5. Optimisation and Numerical Computation
  6. Signal Processing and Algorithmic Geometry

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  ⑥ Signal Processing and Algorithmic Geometry

- other master modules with theory content (Logic and Learning specialization):
  - Program and Resource Analysis (WM 8)
  - Selected Topics in Term Rewriting (WM 9)
  - Introduction to Complexity Theory (WM 20)

Outline

articify with session ID **4957 9500** for anonymous questions

Important Information

- LVA 703302 (VO 2) + 703303 (PS 2)

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Time and Place

VO	Monday	8:15 – 10:00	HSB 9	(AM)
PS	Friday	8:15 – 10:00	SR 12	(SF)

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Consultation Hours

Aart Middeldorp	3M07	Wednesday	11:30 – 13:00
Samuel Frontull	2W04	Tuesday	14:00 – 15:30

Schedule

week 1	06.10 & 10.10	week 6	10.11 & 14.11	week 11	12.01 & 16.01
week 2	13.10	week 7	17.11 & 21.11	week 12	19.01 & 23.01
week 3	20.10 & 24.10	week 8	24.11 & 28.11	week 13	26.01 & 30.01
week 4	27.10 & 31.10	week 9	01.12 & 05.12 & 12.12	week 14	02.02
week 5	03.11 & 07.11	week 10	15.12 & 09.01		

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Grading — Vorlesung

- first exam on February 2

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Grading — Vorlesung

- first exam on February 2
- registration starts 5 weeks and ends 2 weeks before exam

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- second exam on February 26

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- registration starts 5 weeks and ends 2 weeks before exam
- de−registration is possible until 10:00 on January 24
- second exam on February 26
- third exam on September 25 (on demand)

Grading — PS

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grade : $[0,50) \to \mathbf{5}$ $\quad [50,63) \to \mathbf{4}$ $\quad [63,75) \to \mathbf{3}$ $\quad [75,88) \to \mathbf{2}$ $\quad [88,100] \to \mathbf{1}$

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evaluation 24W

Literature

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  Automata and Computability
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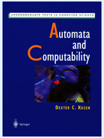

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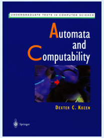

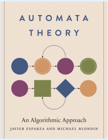

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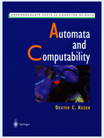

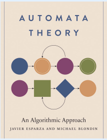

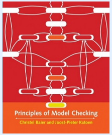

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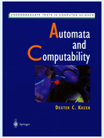

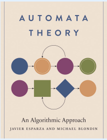

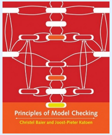

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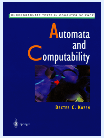

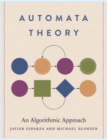

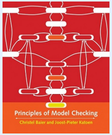

Online Material

- solutions to selected exercises are available after they have been discussed in PS

Outline

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

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- $\{\epsilon, 0, 1, 00, 01, 10, 11\}$ $\quad$ (all strings having at most two symbols)
- $\{x \mid x$ is valid program in some machine language$\}$

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- $\#a(x)$ ($a \in \Sigma$, $x \in \Sigma^*$) denotes number of a's in x

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- power set $\quad 2^A = \{Q \mid Q \subseteq A\}$

Examples

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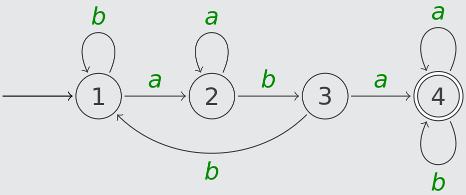
- $\{1, 01\}^* = \{\epsilon, 1, 01, 11, 011, 101, 0101, 111, 0111, 1011, 01011, \dots\}$
- $2^{\{1,01\}} = \{\varnothing, \{1\}, \{01\}, \{1, 01\}\}$

Some Useful Properties

- $\{\epsilon\}A = A\{\epsilon\} = A$
- $\varnothing A = A\varnothing = \varnothing$
- $\sim(A \cup B) = (\sim A) \cap (\sim B)$
- $\sim(A \cap B) = (\sim A) \cup (\sim B)$
- $A^{m+n} = A^m A^n$
- $A^* A^* = A^*$
- $A^{**} = A^*$
- $A^* = \{\epsilon\} \cup AA^* = \{\epsilon\} \cup A^*A$
- $\varnothing^* = \{\epsilon\}$

Outline

Example



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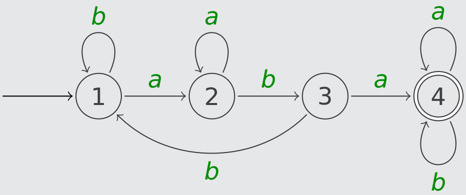

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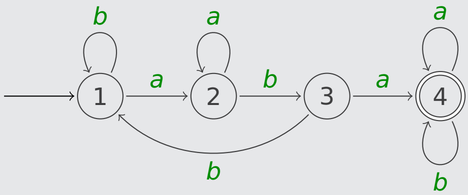

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1	2	1
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  ① Q: finite set of states

  ② Σ: input alphabet

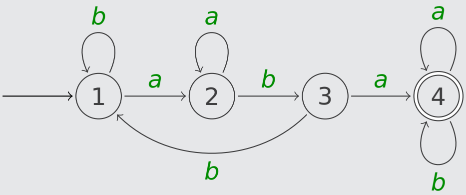
  ③ $\delta\colon Q \times \Sigma \to Q$: transition function

  ④ $s \in Q$: start state

  ⑤ $F \subseteq Q$: final (accept) states

Example

DFA $M = (Q, \Sigma, \delta, s, F)$



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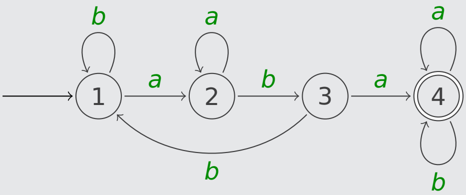

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$b \quad a \quad b \quad a \quad a$

Example

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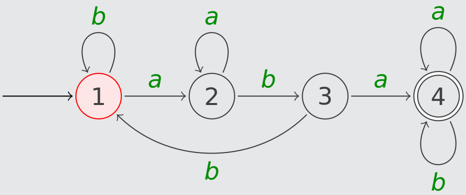

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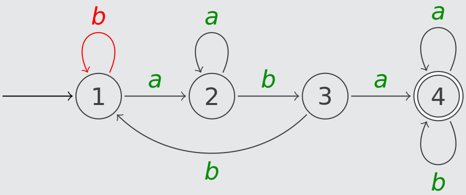

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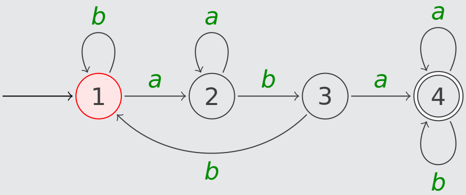

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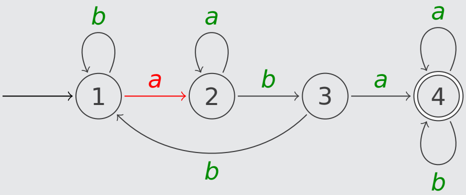

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$1 \quad 1$

Example

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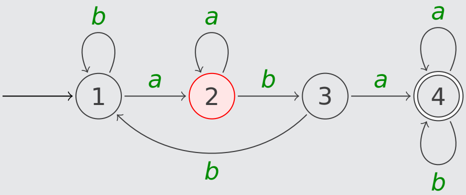

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$b \quad a \quad b \quad a \quad a$

$1 \quad 1 \quad 2$

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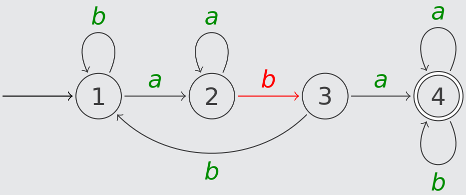

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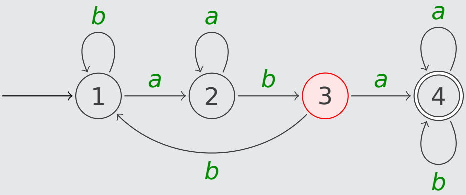

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1 1 2 3

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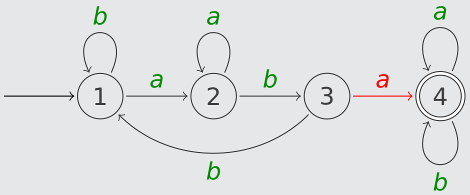

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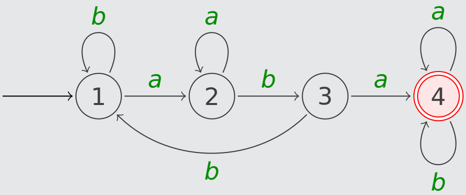

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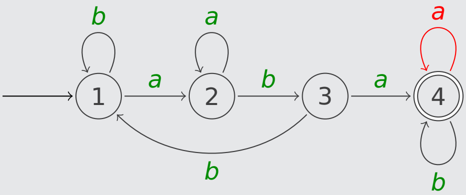

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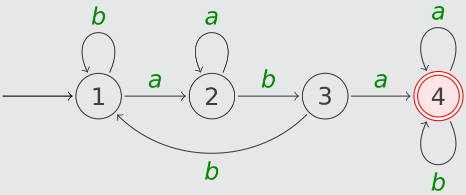

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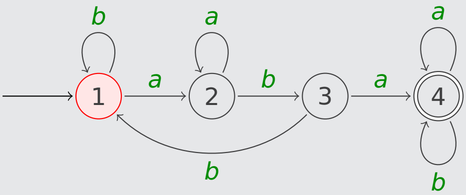

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$a \quad a \quad b \quad b \quad b$
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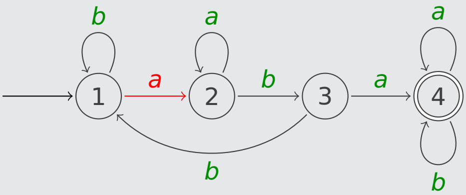

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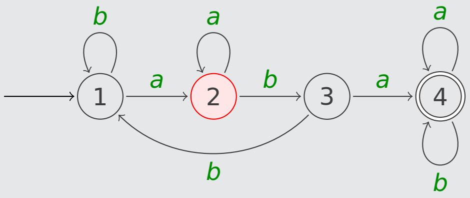

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	b	a	b	a	a
1	1	2	3	4	4

	a	a	b	b	b
1	2				

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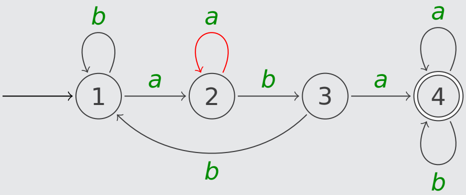

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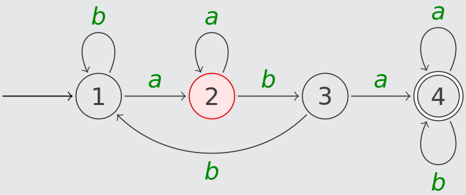

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$$\begin{array}{cccccccccccc} & b & & a & & b & & a & & a & \\ 1 & & 1 & & 2 & & 3 & & 4 & & 4 \end{array}$$

$$\begin{array}{cccccccccccc} & a & & a & & b & & b & & b & \\ 1 & & 2 & & \color{red}{2} & & & & & & \end{array}$$

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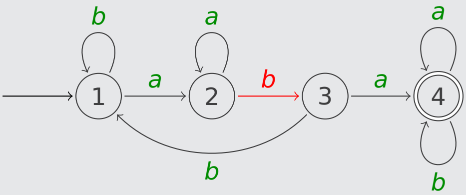

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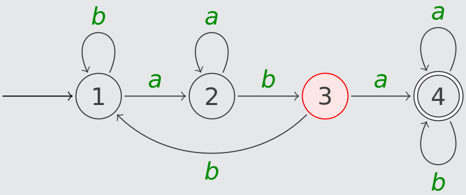
	a	a	b	b	b
1	2	2			

Example

DFA $M = (Q, \Sigma, \delta, s, F)$



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δ	a	b
1	2	1
2	2	3
3	4	1
4	4	4

	b	a	b	a	a
1	1	2	3	4	4

	a	a	b	b	b
1	2	2	3		

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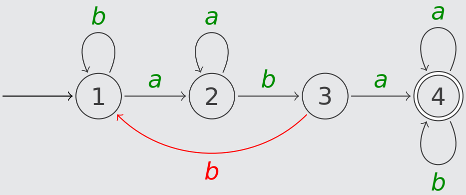

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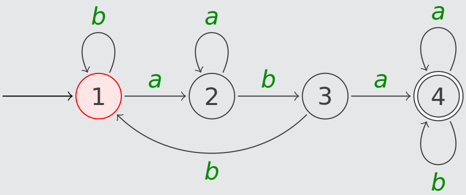

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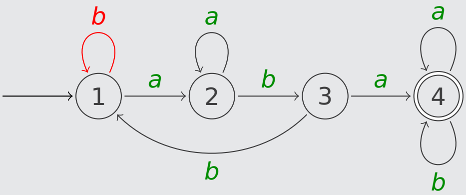

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1	2	2	3	1	

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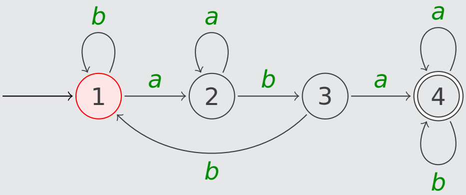

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$\quad b \quad a \quad b \quad a \quad a$
1 1 2 3 4 4

$\quad a \quad a \quad b \quad b \quad b$
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Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\delta\colon Q \times \Sigma \to Q$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\delta}\colon Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \epsilon) = q \qquad \widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$$

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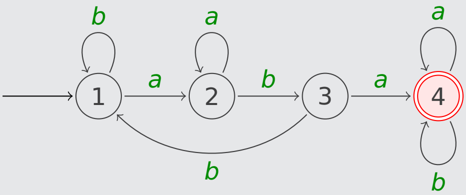
Example

DFA $M = (Q, \Sigma, \delta, s, F)$



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$b \quad a \quad b \quad a \quad a \quad \in L(M)$

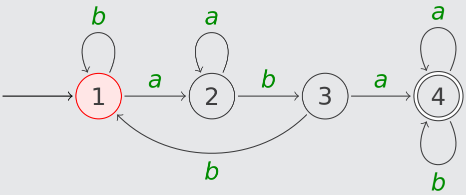
1 1 2 3 4 4

$a \quad a \quad b \quad b \quad b$

1 2 2 3 1 1

Example

DFA $M = (Q, \Sigma, \delta, s, F)$



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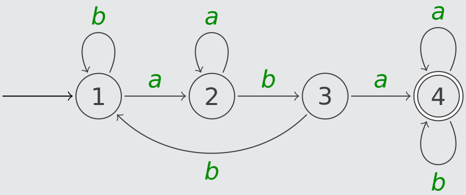

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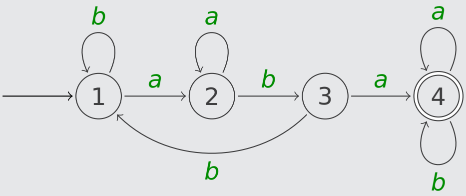
$L(M) = \{x \in \Sigma^* \mid \qquad \}$

Example

DFA $M = (Q, \Sigma, \delta, s, F)$



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$L(M) = \{x \in \Sigma^* \mid x \text{ contains } aba \text{ as substring}\}$

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- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M: $L(M) = \{x \in \Sigma^* \mid \widehat{\delta}(s, x) \in F\}$
- set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Outline

Question

What is the language accepted by the DFA given by the following transition table ?

		a	b
$\rightarrow$	$1\,F$	2	1
	$2\,F$	3	1
	3	3	3

Here the arrow indicates the start state and F marks the final states.

A $\{(ab)^n \mid n \in \mathbb{N}\}$

B $\sim(\{a,b\}^*\{aa\})$

C the set of all strings over $\{a,b\}$ not containing two consecutive a's

D the set of all strings over $\{a,b\}$ with an odd number of a's and an even number of b's

Outline

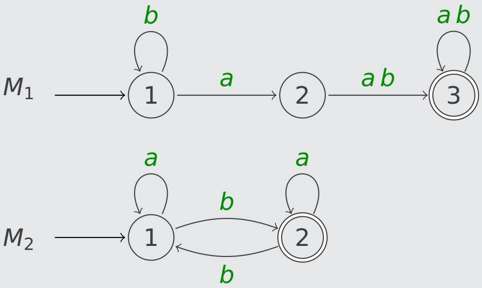
Theorem

regular sets are effectively closed under intersection

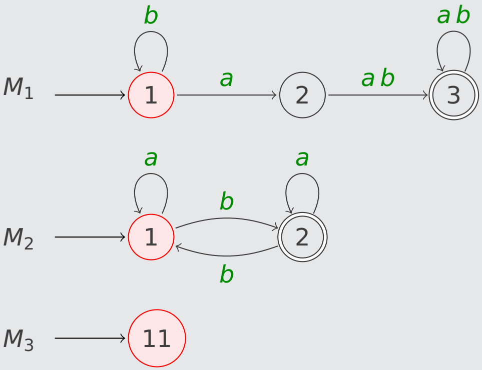
Theorem

regular sets are effectively closed under intersection

Example



Example



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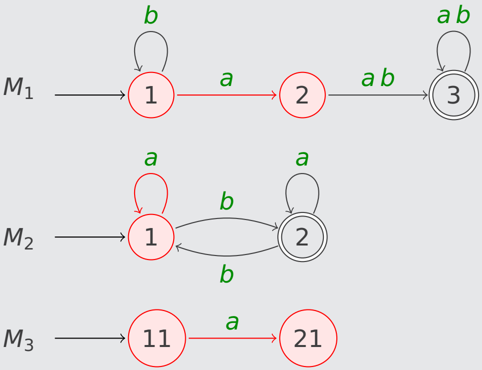

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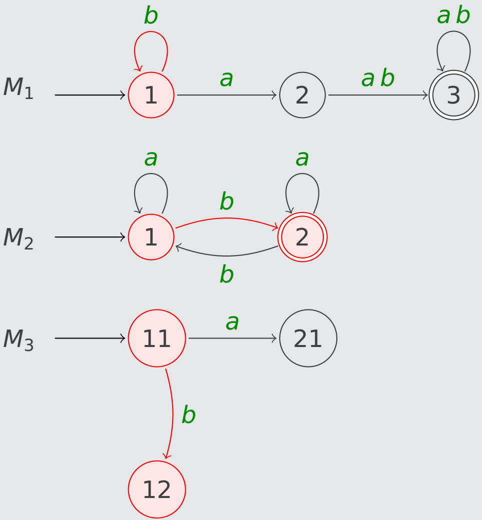
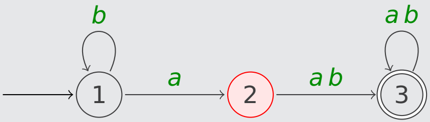
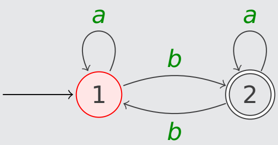


Example

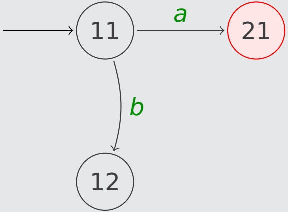
M_1



M_2



M_3



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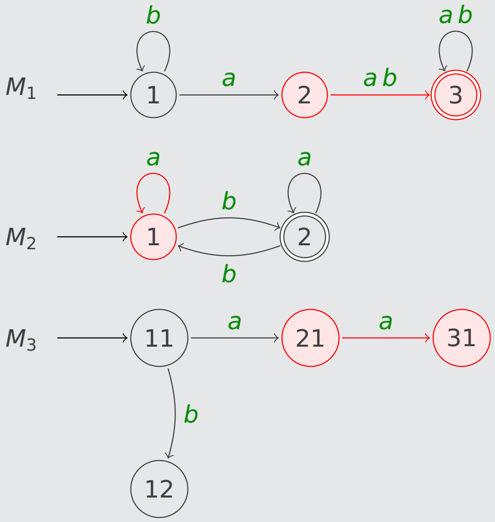

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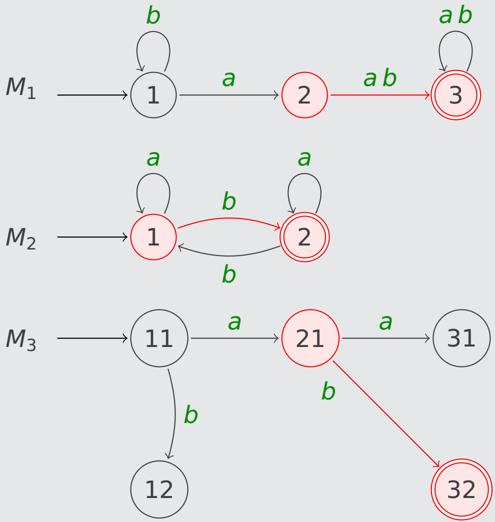

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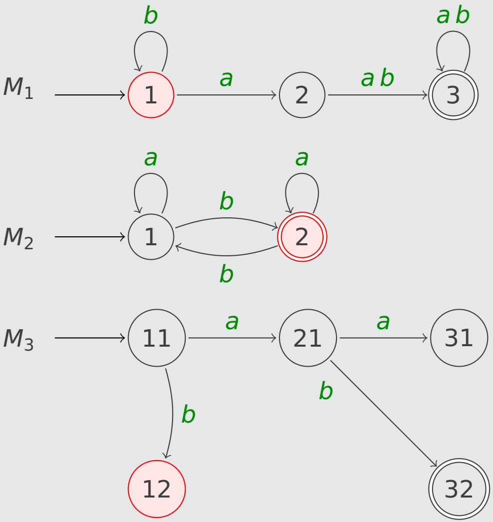

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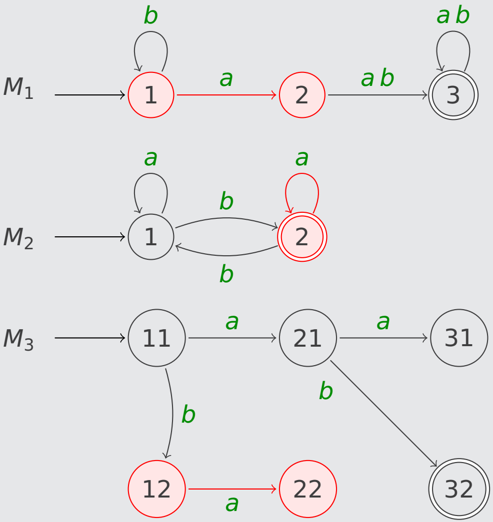

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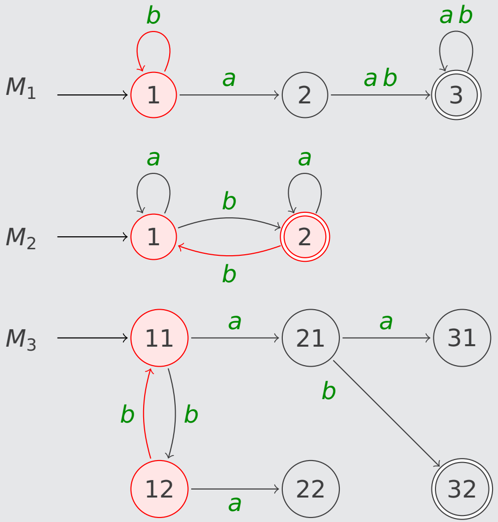

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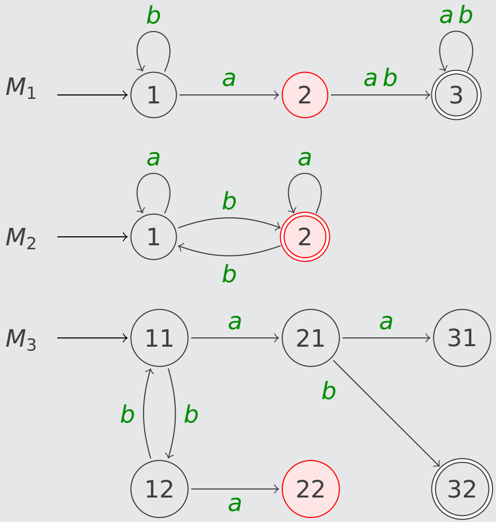

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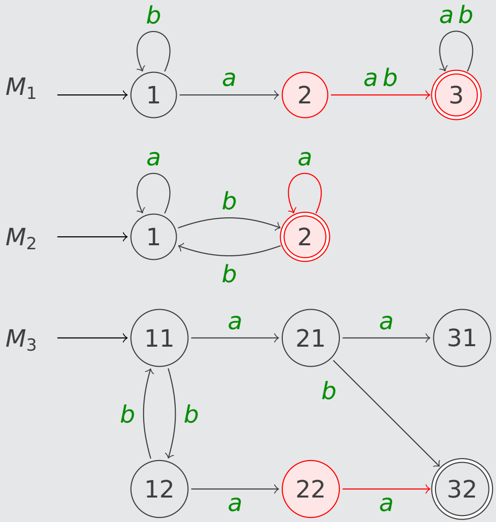

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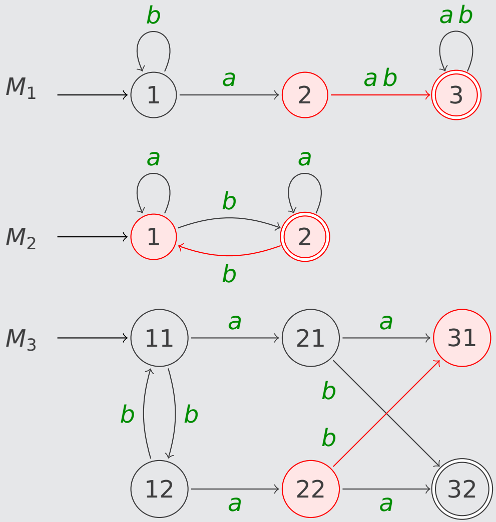

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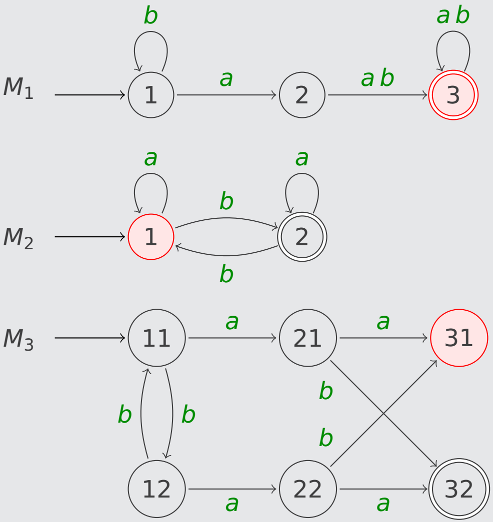

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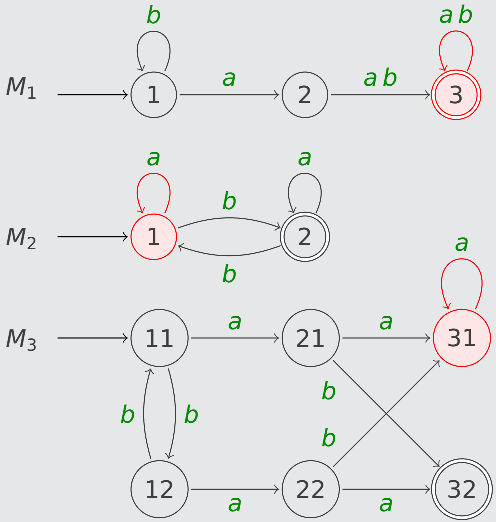

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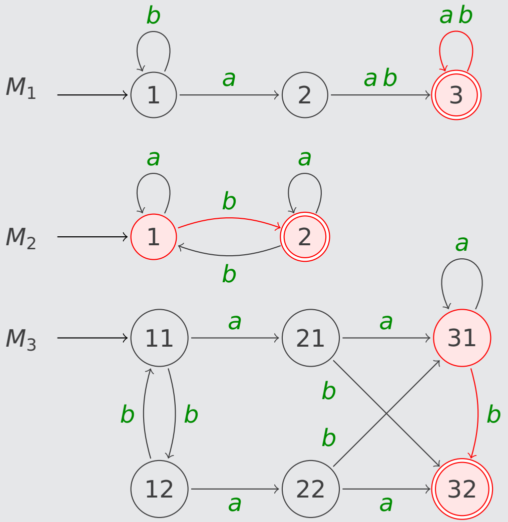

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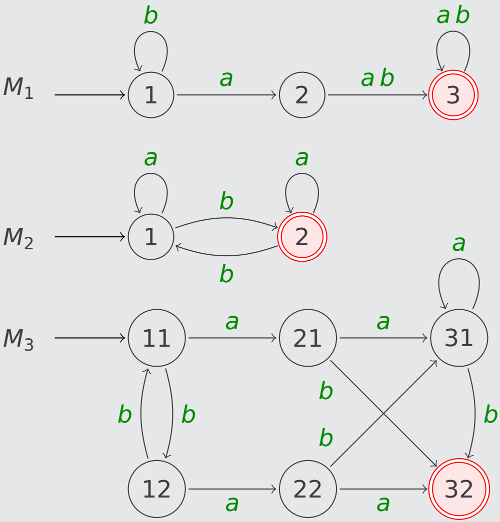

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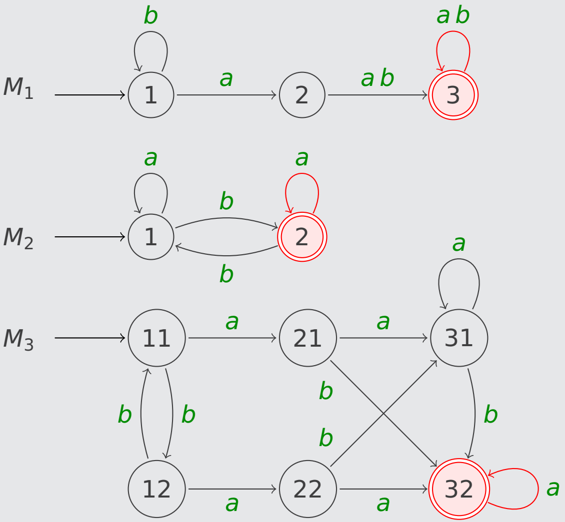

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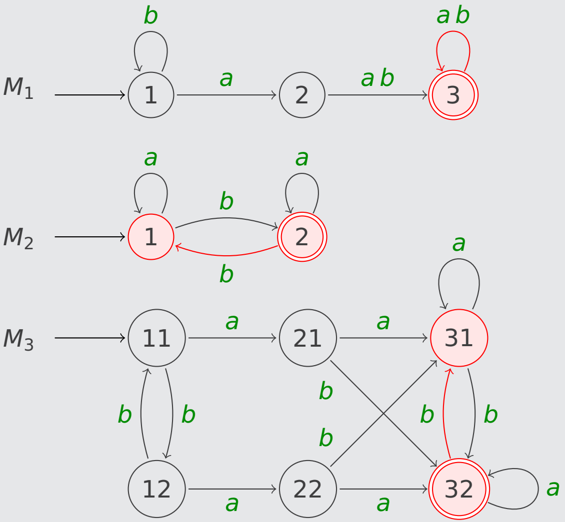

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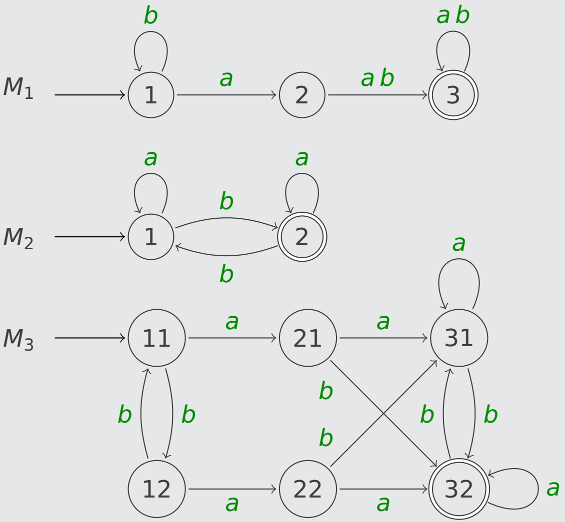

$$L(M_3) = L(M_1) \cap L(M_2)$$

Theorem

regular sets are effectively closed under intersection

Proof

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

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  ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$

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  ② $F_3 = F_1 \times F_2$

  ③ $s_3 = (s_1, s_2)$

  ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for all $p \in Q_1$, $q \in Q_2$, $a \in \Sigma$

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  ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for all $p \in Q_1,\ q \in Q_2,\ a \in \Sigma$
- claim: $\widehat{\delta_3}((p, q), x) = (\widehat{\delta_1}(p, x), \widehat{\delta_2}(q, x))$ for all $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$ (on next slide)

Theorem

regular sets are effectively closed under intersection

Proof (product construction)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- $A \cap B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with
  1. $Q_3 = Q_1 \times Q_2 = \{(p,q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
  2. $F_3 = F_1 \times F_2$
  3. $s_3 = (s_1, s_2)$
  4. $\delta_3((p,q),a) = (\delta_1(p,a), \delta_2(q,a))$ for all $p \in Q_1$, $q \in Q_2$, $a \in \Sigma$
- claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x))$ for all $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$ (on next slide)

Proof of Claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x))$ for all $x \in \Sigma^*$

- base case: $|x| = 0$

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- induction step: $|x| > 0$

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$$\begin{aligned} \widehat{\delta_3}((p,q),x) &= \delta_3(\widehat{\delta_3}((p,q),y),a) && \text{(definition of } \widehat{\delta_3}\text{)} \\ &= \delta_3((\widehat{\delta_1}(p,y), \widehat{\delta_2}(q,y)),a) && \text{(induction hypothesis)} \end{aligned}$$

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&= (\delta_1(\widehat{\delta_1}(p,y),a), \delta_2(\widehat{\delta_2}(q,y),a)) && (\text{definition of } \delta_3) \\
&= (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) && (\text{definition of } \widehat{\delta_1} \text{ and } \widehat{\delta_2})
\end{aligned}$$

Theorem

regular sets are effectively closed under complement

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Proof

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

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  1. $Q_2 = Q_1$
  2. $\delta_2(q, a) = \delta_1(q, a)$ for all $q \in Q_2$ and $a \in \Sigma$
  3. $s_2 = s_1$

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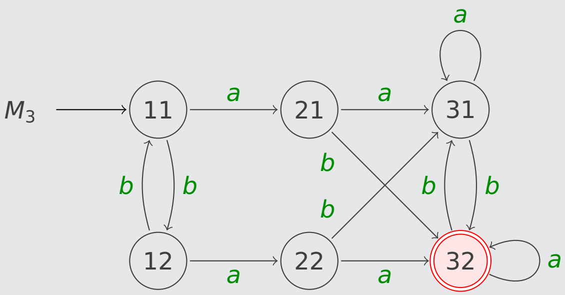
  ③ $s_2 = s_1$

  ④ $F_2 = Q_1 - F_1$

Example



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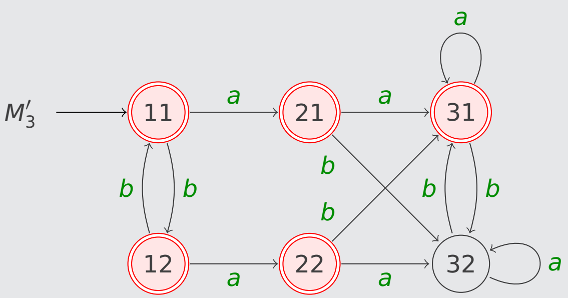

$L(M_3') = \sim L(M_3)$

Theorem

regular sets are effectively closed under union

Theorem

regular sets are effectively closed under union

Proof

$$A \cup B = \sim((\sim A) \cap (\sim B))$$

Theorem

regular sets are effectively closed under union

Proof (explicit construction)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$

  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

- $A \cup B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with

  ① $Q_3 = Q_1 \times Q_2 = \{(p,q) \mid p \in Q_1 \text{ and } q \in Q_2\}$

  ② $F_3 = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

  ③ $s_3 = (s_1, s_2)$

  ④ $\delta_3((p,q),a) = (\delta_1(p,a), \delta_2(q,a))$ for all $p \in Q_1$, $q \in Q_2$, $a \in \Sigma$

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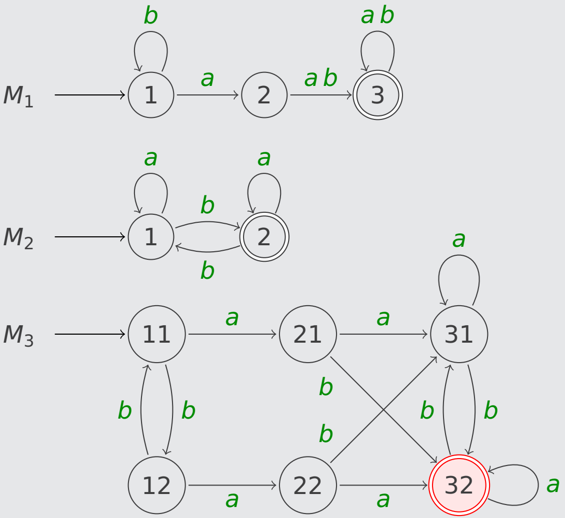

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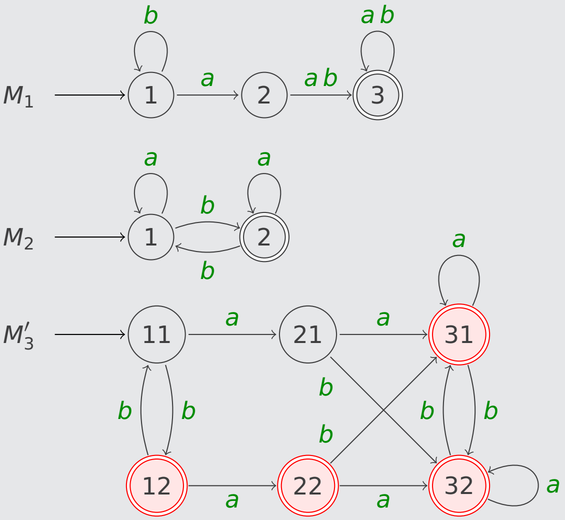

$L(M_3') = L(M_1) \cup L(M_2)$

Outline

Kozen

- Lectures 1 – 4

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Important Concepts

- alphabet
- closure properties
- DFA
- language
- product construction
- regular set
- string

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Solutions

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... bonus exercises give bonus points