



Automata and Logic

Aart Middeldorp and Samuel Frontull

Initial Remarks

- ▶ **Automata and Logic** is elective module 1 in master program Computer Science
- ▶ master students must select 3 out of 6 elective modules:
 - ① Automata and Logic
 - ② Constraint Solving (offered in 25S)
 - ③ Cryptography (offered in 25S)
 - ④ High-Performance Computing
 - ⑤ Optimisation and Numerical Computation
 - ⑥ Signal Processing and Algorithmic Geometry
- ▶ other master modules with theory content (**Logic and Learning** specialization):
 - ▶ Program and Resource Analysis (WM 8)
 - ▶ Selected Topics in Term Rewriting (WM 9)
 - ▶ Introduction to Complexity Theory (WM 20)

Outline

1. Introduction

Organisation

Contents

2. Basic Definitions

3. Deterministic Finite Automata

4. Intermezzo

5. Closure Properties

6. Further Reading

 with session ID **4957 9500** for anonymous questions



Important Information

- ▶ LVA 703302 (VO 2) + 703303 (PS 2)
- ▶ <http://cl-informatik.uibk.ac.at/teaching/ws25/al>
- ▶ online registration for VO required
- ▶ OLAT links for VO and PS

Time and Place

VO	Monday	8:15–10:00	HSB 9	(AM)
PS	Friday	8:15–10:00	SR 12	(SF)

Consultation Hours

Aart Middeldorp	3M07	Wednesday	11:30–13:00
Samuel Frontull	2W04	Tuesday	14:00–15:30

Schedule

week 1	06.10 & 10.10	week 6	10.11 & 14.11	week 11	12.01 & 16.01
week 2	13.10	week 7	17.11 & 21.11	week 12	19.01 & 23.01
week 3	20.10 & 24.10	week 8	24.11 & 28.11	week 13	26.01 & 30.01
week 4	27.10 & 31.10	week 9	01.12 & 05.12 & 12.12	week 14	02.02
week 5	03.11 & 07.11	week 10	15.12 & 09.01		

Grading — Vorlesung

- ▶ first exam on February 2
- ▶ registration starts 5 weeks and ends 2 weeks before exam
- ▶ de-registration is possible until 10:00 on January 24
- ▶ second exam on February 26
- ▶ third exam on September 25 (on demand)

score = $\min(\frac{2}{3}(E + P) + B, 100)$ E : points for solved **exercises** (at most 130)

B : points for **bonus exercises** (at most 20)

P : points for **presentations** of solutions (at most 20)

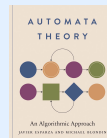
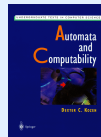
grade : $[0, 50) \rightarrow \mathbf{5}$ $[50, 63) \rightarrow \mathbf{4}$ $[63, 75) \rightarrow \mathbf{3}$ $[75, 88) \rightarrow \mathbf{2}$ $[88, 100] \rightarrow \mathbf{1}$

- ▶ homework exercises are given on course web site
- ▶ solved exercises must be marked and solutions must be uploaded (**PDF**) in **OLAT**
- ▶ strict deadline: 7 am on Friday
- ▶ 10 points per PS
- ▶ two presentations of solutions are mandatory
- ▶ 20 points for two presentations; additional presentations give bonus points
- ▶ attendance is compulsory; unexcused absence is allowed twice (resulting in 0 points)

evaluation 24W

Literature

- ▶ Dexter C Kozen
Automata and Computability
Springer-Verlag, 1997
- ▶ Javier Esparza and Michael Blondin
Automata Theory: An Algorithmic Approach
MIT Press, 2023
- ▶ Christel Baier and Joost-Pieter Katoen
Principles of Model Checking
MIT Press, 2008
- ▶ additional resources will be linked from course website



Online Material

- ▶ solutions to selected exercises are available after they have been discussed in PS

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Automata

- ▶ (**deterministic**, nondeterministic, alternating) **finite automata**
- ▶ regular expressions
- ▶ (alternating) Büchi automata

Logic

- ▶ (weak) monadic second-order logic
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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Definitions

- ▶ **alphabet** is finite set; its elements are called **symbols** or **letters**
- ▶ **string** over alphabet Σ is finite sequence of elements of Σ
- ▶ **length** $|x|$ of string x is number of symbols in x
- ▶ **empty string** is unique string of length 0 and denoted by ϵ
- ▶ Σ^* is set of all strings over Σ ($\emptyset^* = \{\epsilon\}$)
- ▶ **language** over Σ is subset of Σ^*

Examples

strings over $\Sigma = \{0, 1\}$: 0 0110 ϵ

languages over Σ :

- ▶ $\{\epsilon, 0, 1, 00, 01, 10, 11\}$ (all strings having at most two symbols)
- ▶ $\{x \mid x \text{ is valid program in some machine language}\}$

Definitions

- ▶ **string concatenation** $x, y \in \Sigma^* \implies xy \in \Sigma^*$ is associative:

$$(xy)z = x(yz) \quad \text{for all } x, y, z \in \Sigma^*$$

- ▶ empty string is **identity** for concatenation:

$$\epsilon x = x\epsilon = x \quad \text{for all } x \in \Sigma^*$$

- ▶ x is **substring** (**prefix**, **suffix**) of y if $y = uxv$ ($y = xv$, $y = ux$)

- ▶ x^n ($x \in \Sigma^*$, $n \in \mathbb{N}$):

$$x^0 = \epsilon$$

$$x^{n+1} = x^n x$$

- ▶ $\#a(x)$ ($a \in \Sigma$, $x \in \Sigma^*$) denotes number of a 's in x

Definitions

for $A, B \subseteq \Sigma^*$

► **union**

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

► **intersection**

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

► **complement**

$$\sim A = \Sigma^* - A = \{x \in \Sigma^* \mid x \notin A\}$$

► **set concatenation**

$$AB = \{xy \mid x \in A \text{ and } y \in B\}$$

► **powers** A^n ($n \in \mathbb{N}$)

$$A^0 = \{\epsilon\} \quad A^{n+1} = AA^n$$

► **asterate** A^* is union of all finite powers of A

$$A^* = \bigcup_{n \geq 0} A^n = \{x_1 x_2 \cdots x_n \mid n \geq 0 \text{ and } x_i \in A \text{ for all } 1 \leq i \leq n\}$$

► $A^+ = AA^* = \bigcup_{n \geq 1} A^n$

► **power set**

$$2^A = \{Q \mid Q \subseteq A\}$$

Examples

- ▶ substrings of 011: 0, 1, 01, 11, 011, ϵ
- ▶ prefixes of 011: 0, 01, 011, ϵ
- ▶ suffixes of 011: 1, 11, 011, ϵ
- ▶ $(011)^3 = 011011011 \neq 011^3$
- ▶ $\#1(011011011) = 6$ $\#0(\epsilon) = 0$
- ▶ $\{0, 10, 111\}\{1, 11\} = \{01, 101, 1111, 011, 1011, 11111\}$
- ▶ $\{0, 01, 111\}\{1, 11\} = \{01, 011, 1111, 0111, 11111\}$
- ▶ $\{1, 01\}^3 = \{111, 0111, 1011, 01011, 1101, 01101, 10101, 010101\}$
- ▶ $\{1, 01\}^* = \{\epsilon, 1, 01, 11, 011, 101, 0101, 111, 0111, 1011, 01011, \dots\}$
- ▶ $2^{\{1, 01\}} = \{\emptyset, \{1\}, \{01\}, \{1, 01\}\}$

Some Useful Properties

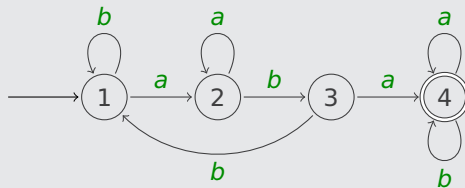
- ▶ $\{\epsilon\}A = A\{\epsilon\} = A$
- ▶ $\emptyset A = A\emptyset = \emptyset$
- ▶ $\sim(A \cup B) = (\sim A) \cap (\sim B)$
- ▶ $\sim(A \cap B) = (\sim A) \cup (\sim B)$
- ▶ $A^{m+n} = A^m A^n$
- ▶ $A^* A^* = A^*$
- ▶ $A^{**} = A^*$
- ▶ $A^* = \{\epsilon\} \cup AA^* = \{\epsilon\} \cup A^*A$
- ▶ $\emptyset^* = \{\epsilon\}$

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Example

DFA $M = (Q, \Sigma, \delta, s, F)$



1 $Q = \{1, 2, 3, 4\}$

2 $\Sigma = \{a, b\}$

3 $\delta: Q \times \Sigma \rightarrow Q$

4 $s = 1$

5 $F = \{4\}$

δ	a	b
1	2	1
2	2	3
3	4	1
4	4	4

$b \ a \ b \ a \ a \in L(M)$
1 1 2 3 4 4

$a \ a \ b \ b \ b \notin L(M)$
1 2 2 3 1 1

$L(M) = \{x \in \Sigma^* \mid x \text{ contains } aba \text{ as substring}\}$

Definitions

► **deterministic finite automaton (DFA)** is quintuple $M = (Q, \Sigma, \delta, s, F)$ with

- ① Q : finite set of **states**
- ② Σ : **input alphabet**
- ③ $\delta: Q \times \Sigma \rightarrow Q$: **transition function**
- ④ $s \in Q$: **start** state
- ⑤ $F \subseteq Q$: **final (accept) states**

► $\hat{\delta}: Q \times \Sigma^* \rightarrow Q$ is inductively defined by

$$\hat{\delta}(q, \epsilon) = q$$

$$\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is **accepted** by M if $\hat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is **rejected** by M if $\hat{\delta}(s, x) \notin F$
- language accepted by M : $L(M) = \{x \in \Sigma^* \mid \hat{\delta}(s, x) \in F\}$
- set $A \subseteq \Sigma^*$ is **regular** if $A = L(M)$ for some DFA M

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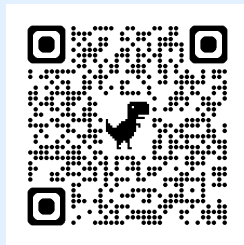
Question

What is the language accepted by the DFA given by the following transition table ?

		<i>a</i>	<i>b</i>
→	1 <i>F</i>	2	1
	2 <i>F</i>	3	1
	3	3	3

Here the arrow indicates the start state and *F* marks the final states.

- A** $\{(ab)^n \mid n \in \mathbb{N}\}$
- B** $\sim(\{a, b\}^* \{aa\})$
- C** the set of all strings over $\{a, b\}$ not containing two consecutive *a*'s
- D** the set of all strings over $\{a, b\}$ with an odd number of *a*'s and an even number of *b*'s



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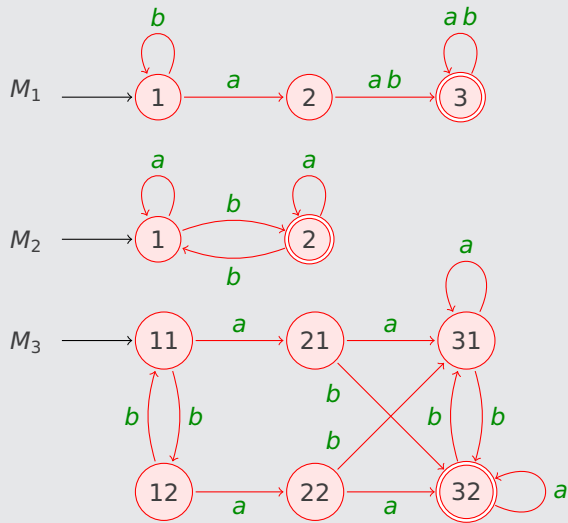
Theorem

regular sets are **effectively** closed under **intersection**

Proof (product construction)

- ▶ $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
 $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- ▶ $A \cap B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with
 - ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
 - ② $F_3 = F_1 \times F_2$
 - ③ $s_3 = (s_1, s_2)$
 - ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for all $p \in Q_1, q \in Q_2, a \in \Sigma$
- ▶ claim: $\hat{\delta}_3((p, q), x) = (\hat{\delta}_1(p, x), \hat{\delta}_2(q, x))$ for all $x \in \Sigma^*$
proof of claim: easy induction on $|x|$ (on next slide)

Example



$$L(M_3) = L(M_1) \cap L(M_2)$$

Proof of Claim

claim: $\widehat{\delta}_3((p, q), x) = (\widehat{\delta}_1(p, x), \widehat{\delta}_2(q, x))$ for all $x \in \Sigma^*$

► base case: $|x| = 0$ and thus $x = \epsilon$

$$\widehat{\delta}_3((p, q), x) = (p, q) = (\widehat{\delta}_1(p, x), \widehat{\delta}_2(q, x))$$

► induction step: $|x| > 0$ and thus $x = ya$ with $|y| = |x| - 1$

$$\begin{aligned}\widehat{\delta}_3((p, q), x) &= \delta_3(\widehat{\delta}_3((p, q), y), a) && \text{(definition of } \widehat{\delta}_3) \\ &= \delta_3((\widehat{\delta}_1(p, y), \widehat{\delta}_2(q, y)), a) && \text{(induction hypothesis)} \\ &= (\delta_1(\widehat{\delta}_1(p, y), a), \delta_2(\widehat{\delta}_2(q, y), a)) && \text{(definition of } \delta_3) \\ &= (\widehat{\delta}_1(p, x), \widehat{\delta}_2(q, x)) && \text{(definition of } \widehat{\delta}_1 \text{ and } \widehat{\delta}_2)\end{aligned}$$

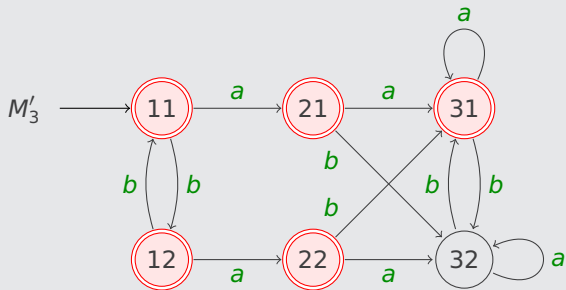
Theorem

regular sets are **effectively** closed under **complement**

Proof

- ▶ $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
- ▶ $\sim A = \Sigma^* - A = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ with
 - ① $Q_2 = Q_1$
 - ② $\delta_2(q, a) = \delta_1(q, a)$ for all $q \in Q_2$ and $a \in \Sigma$
 - ③ $s_2 = s_1$
 - ④ $F_2 = Q_1 - F_1$

Example



$$L(M'_3) = \sim L(M_3)$$

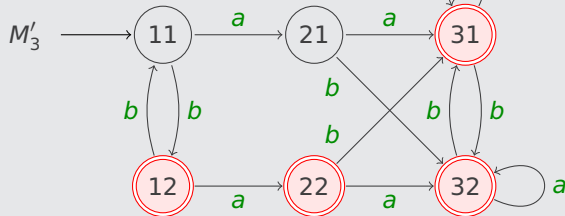
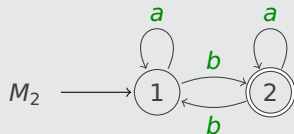
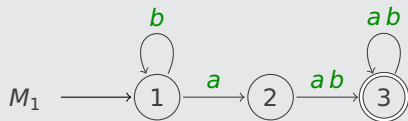
Theorem

regular sets are **effectively** closed under **union**

Proof (explicit construction)

- ▶ $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
 $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- ▶ $A \cup B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with
 - ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
 - ② $F_3 = (F_1 \times Q_2) \cup (Q_1 \times F_2)$
 - ③ $s_3 = (s_1, s_2)$
 - ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for all $p \in Q_1, q \in Q_2, a \in \Sigma$

Example



$$L(M'_3) = L(M_1) \cup L(M_2)$$

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- ▶ Lectures 1–4

Important Concepts

- ▶ alphabet
- ▶ closure properties
- ▶ DFA
- ▶ language
- ▶ product construction
- ▶ regular set
- ▶ string

homework for October 10

Solutions

- ... must be uploaded (PDF format) in OLAT **before 7 am on Friday**
- ... bonus exercises give **bonus points**