Automata and Logic

Aart Middeldorp and Samuel Frontull

Outline

1. **Summary of Previous Lecture**
2. **Nondeterministic Finite Automata**
3. **Epsilon Transitions**
4. **Intermezzo**
5. **Closure Properties**
6. **Hamming Distance**
7. **Further Reading**

Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\delta \colon Q \times \Sigma \to Q$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\delta} \colon Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \epsilon) = q \qquad \widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is accepted by M if $\widehat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M: $L(M) = \{x \in \Sigma^* \mid \widehat{\delta}(s, x) \in F\}$

Definition

set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Theorem

regular sets are effectively closed under intersection, union, and complement

Automata

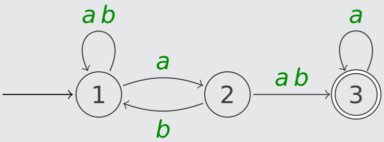
- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second−order logic
- Presburger arithmetic
- linear−time temporal logic

Outline

Example



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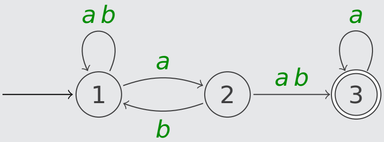
Example

NFA $M = (Q, \Sigma, \Delta, S, F)$



1. $Q = \{1, 2, 3\}$
2. $\Sigma = \{a, b\}$
3. $\Delta : Q \times \Sigma \to 2^Q$
4. $S = \{1\}$
5. $F = \{3\}$

Δ	a	b
1	$\{1, 2\}$	$\{1\}$
2	$\{3\}$	$\{1, 3\}$
3	$\{3\}$	$\varnothing$

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- $\widehat{\Delta} \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

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- $L(N) = \{x \in \Sigma^* \mid \widehat{\Delta}(S, x) \cap F \neq \varnothing\}$

Theorem

every set accepted by NFA is regular

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- claim: $\widehat{\delta_M}(A, x) = \widehat{\Delta}_N(A, x)$ for all $A \subseteq Q_N$ and $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$

Theorem

every set accepted by NFA is regular

Proof (subset construction)

- NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- $L(N) = L(M)$ for DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ with
  1. $Q_M = 2^{Q_N}$
  2. $\delta_M(A, a) = \widehat{\Delta}_N(A, a)$ for all $A \subseteq Q_N$ and $a \in \Sigma$
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  4. $F_M = \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\}$
- claim: $\widehat{\delta_M}(A, x) = \widehat{\Delta}_N(A, x)$ for all $A \subseteq Q_N$ and $x \in \Sigma^*$

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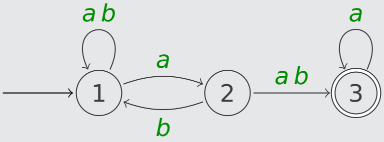

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$B = \{1\}$ $\quad F = \{1,3\}$

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$D = \{3\}$ $\quad H = \{1,2,3\}$

	a	b
A		
B		
C		
D		

	a	b
E		
F		
G		
H		

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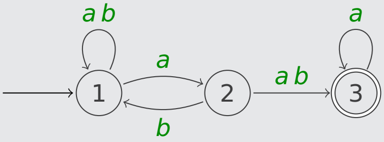

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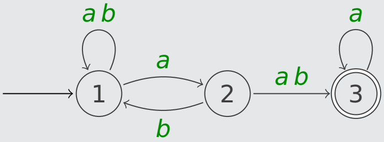

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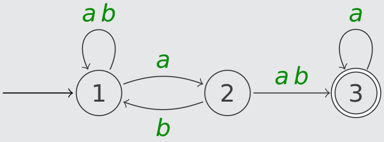

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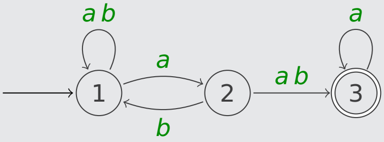

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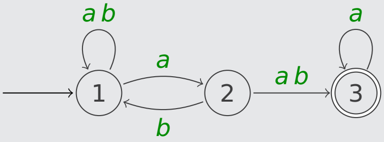

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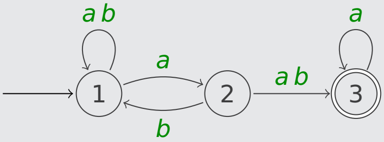

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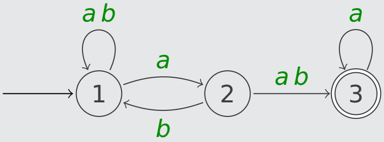

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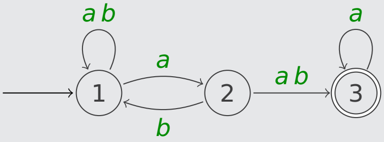

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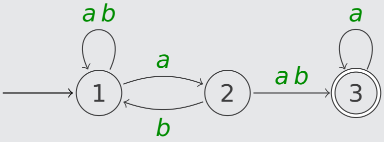

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	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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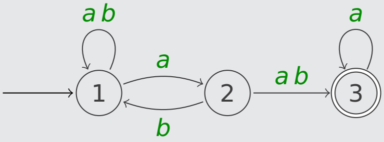

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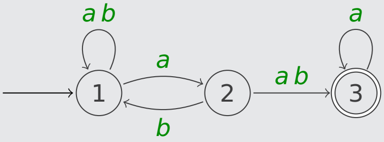

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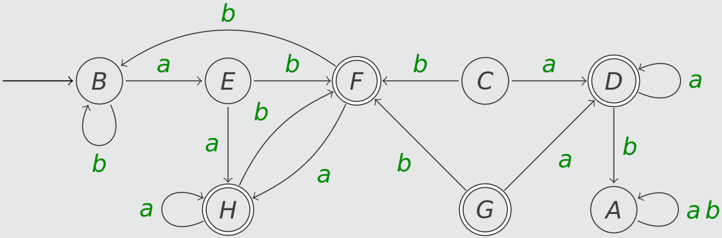

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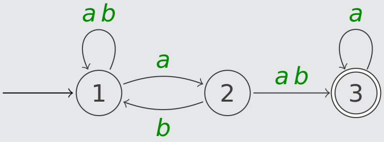

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	a	*b*
A	*A*	*A*
B	*E*	*B*
C	*D*	*F*
D	*D*	*A*

	a	*b*
E	*H*	*F*
F	*H*	*B*
G	*D*	*F*
H	*H*	*F*

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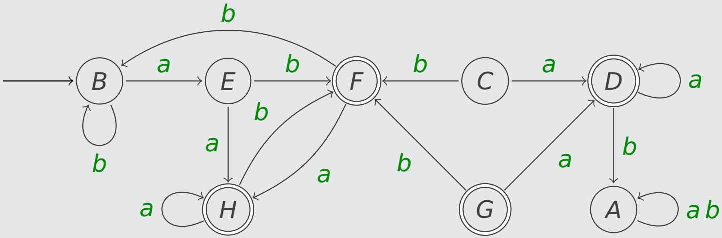

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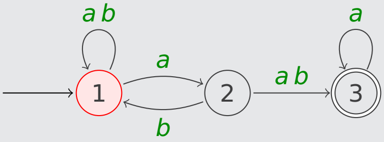

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	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
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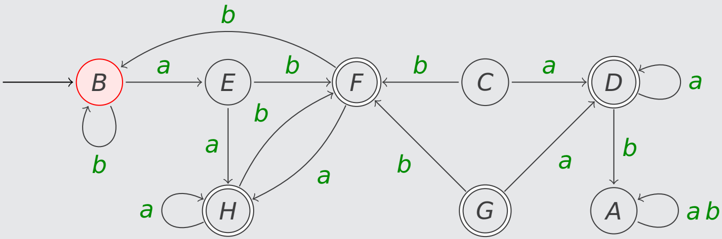
abbbaababbabbbaababba



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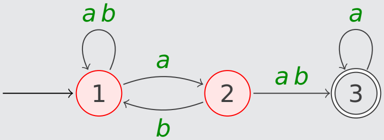

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	a	b
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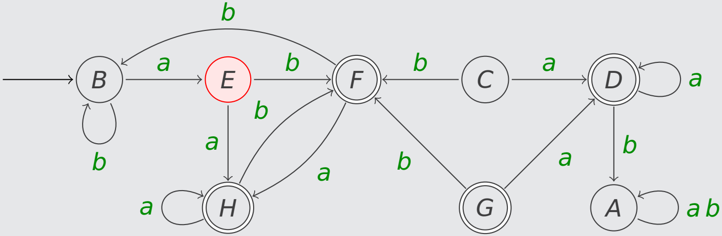
abbbaababbabbbaababba



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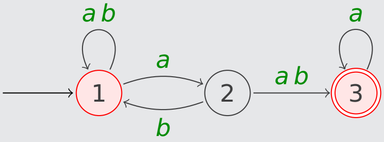

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	a	b
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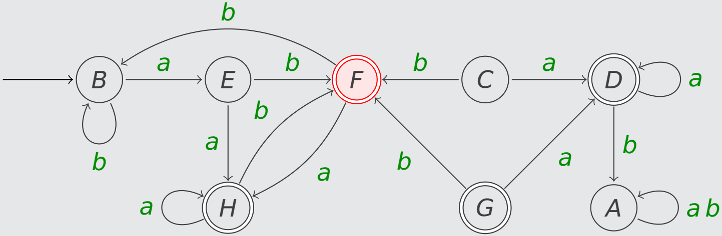

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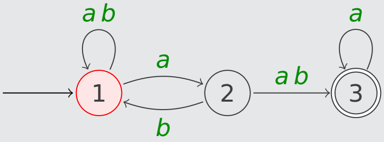

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	a	b
E	H	F
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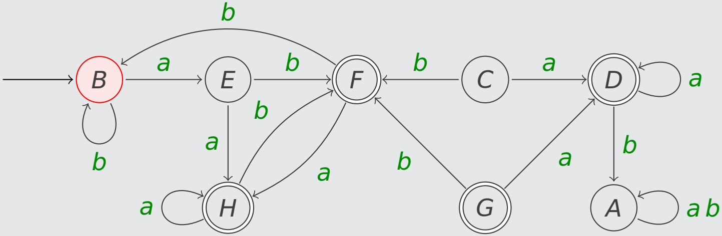
abbbaababbabbbaababba



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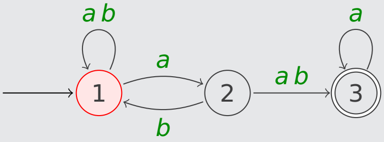

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	a	b
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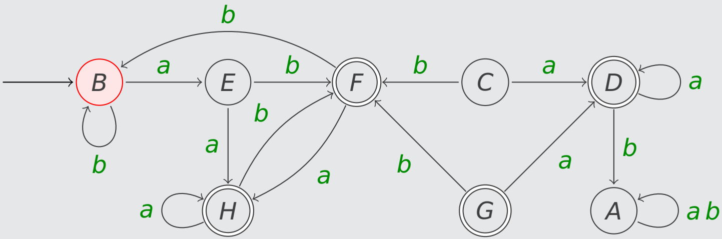
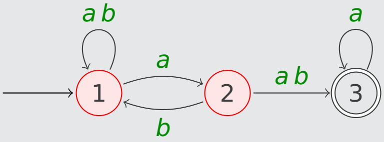
H	H	F

abbbaababbabbbaababba



Example

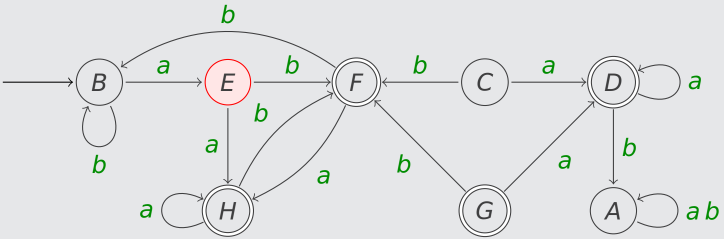


$$A = \varnothing \qquad E = \{1,2\}$$
$$B = \{1\} \qquad F = \{1,3\}$$
$$C = \{2\} \qquad G = \{2,3\}$$
$$D = \{3\} \qquad H = \{1,2,3\}$$

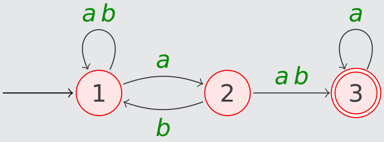
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaabababbabbbaababba



Example



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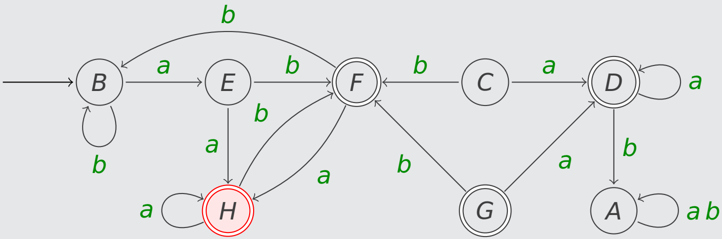
$$D = \{3\} \quad H = \{1,2,3\}$$

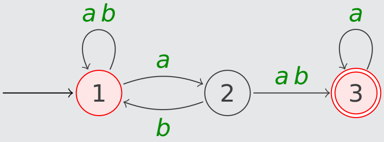
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$$
\begin{array}{llll}
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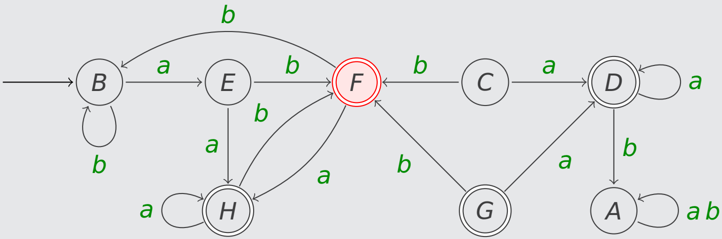
\end{array}
$$

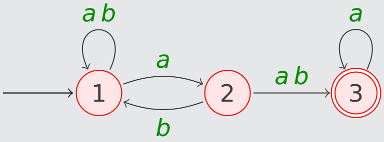
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example

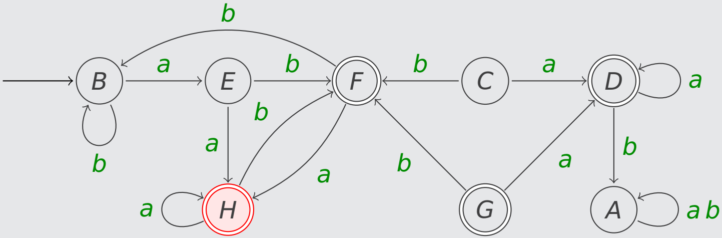


$$A = \varnothing \quad B = \{1\} \quad C = \{2\} \quad D = \{3\}$$
$$E = \{1,2\} \quad F = \{1,3\} \quad G = \{2,3\} \quad H = \{1,2,3\}$$

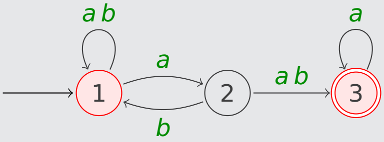
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing$ $\quad E = \{1,2\}$

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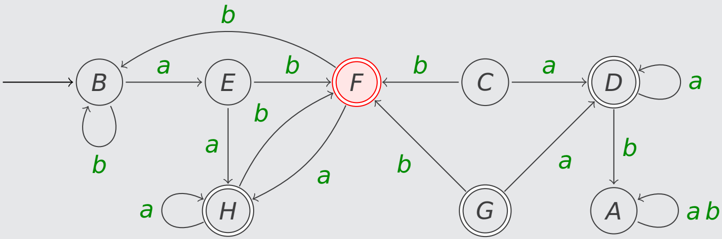
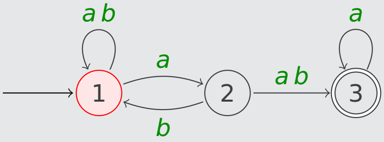
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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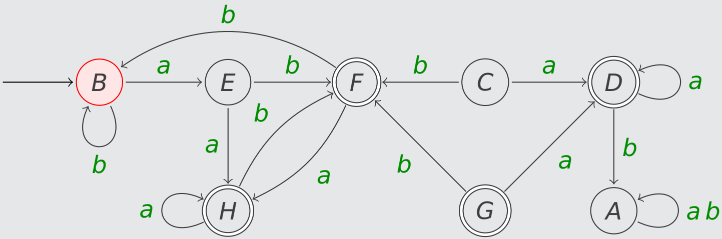
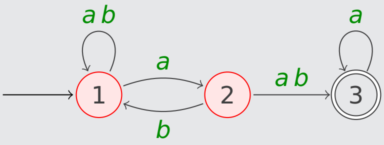
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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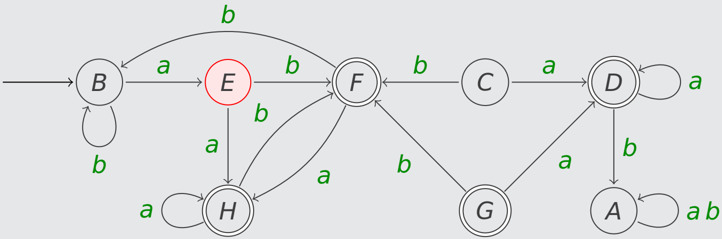
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



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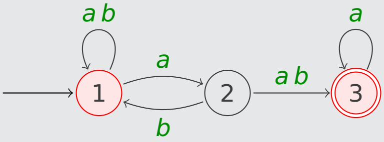

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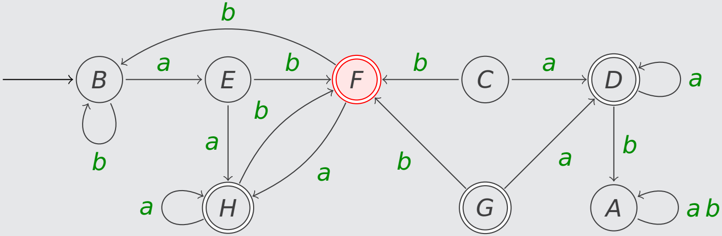
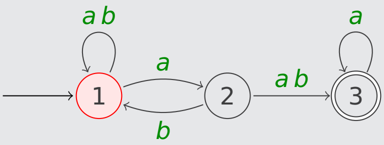
abbbaababbabbbaababba



Example



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	a	b
A	A	A
B	E	B
C	D	F
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	a	b
E	H	F
F	H	B
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H	H	F

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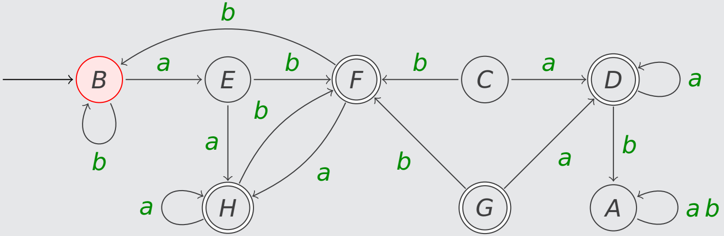
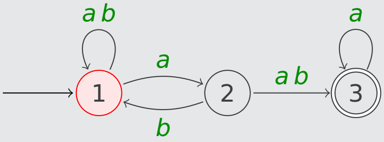


Example



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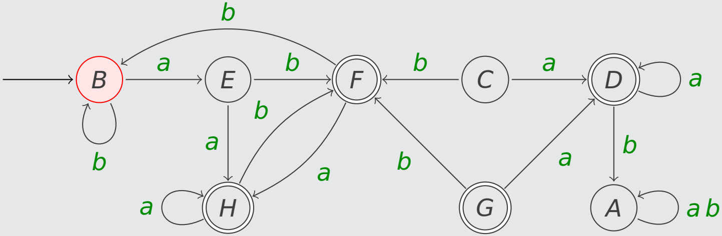
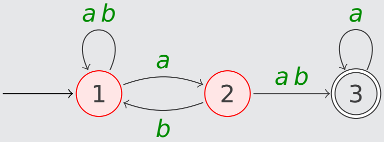


Example



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	a	b
E	H	F
F	H	B
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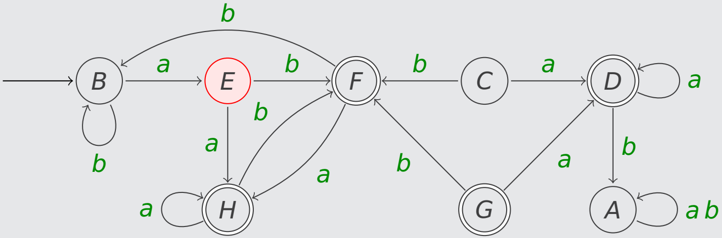
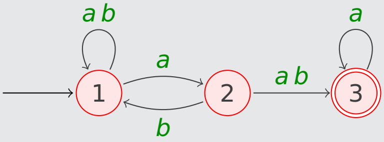
abbbaababbabbbaababba



Example



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	a	b
E	H	F
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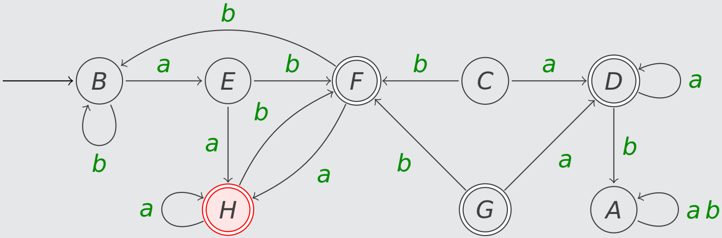
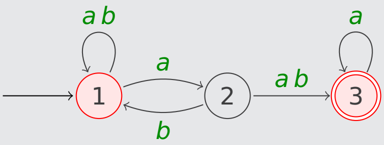


Example



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E	H	F
F	H	B
G	D	F
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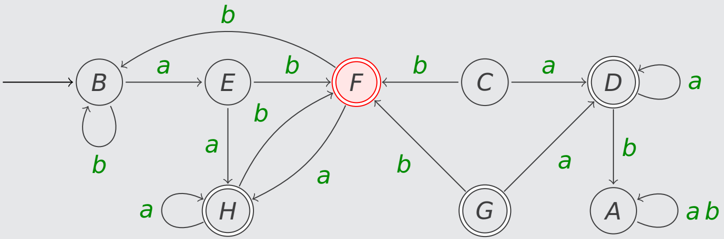
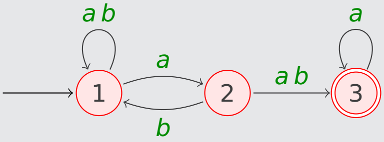


Example



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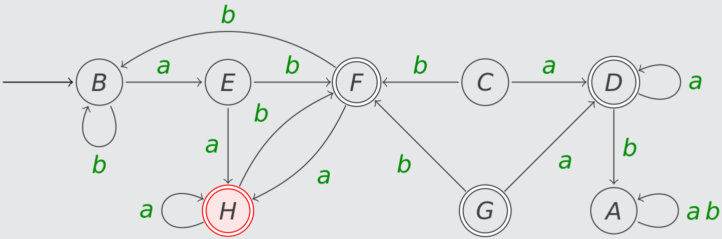
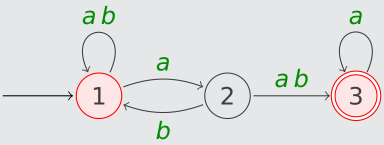
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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	a	b
A	A	A
B	E	B
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	a	b
E	H	F
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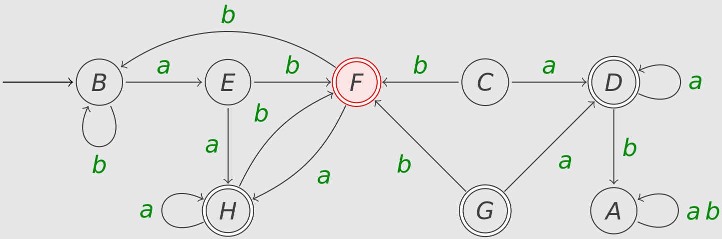
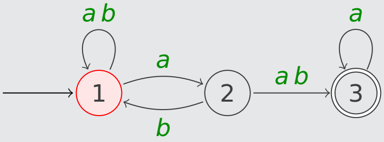
abbbaababbabbbaababba



Example



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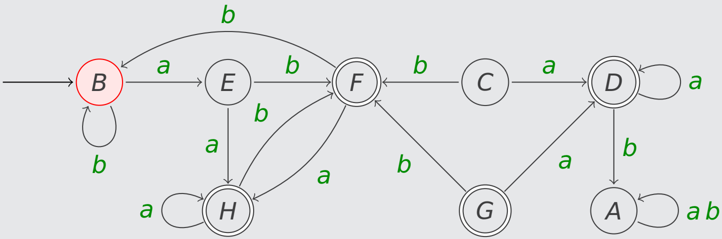
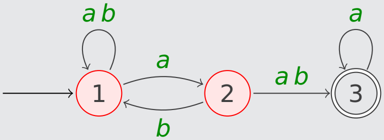
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$$
\begin{array}{lll}
A = \varnothing & \quad & E = \{1,2\} \\
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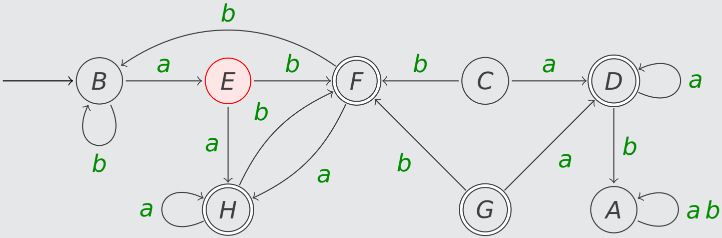
\end{array}
$$

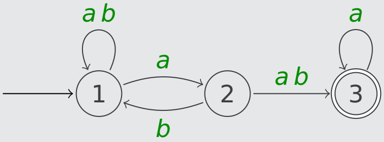
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



$A = \varnothing \quad E = \{1,2\}$

$B = \{1\} \quad F = \{1,3\}$

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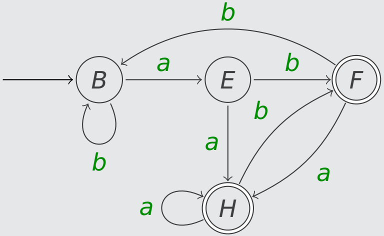
	a	b
A	A	A
B	E	B
C	D	F
D	D	A

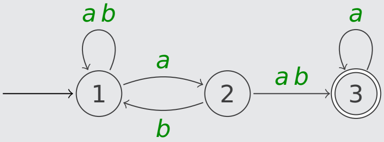
	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba

remove inaccesible states



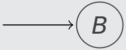
Example



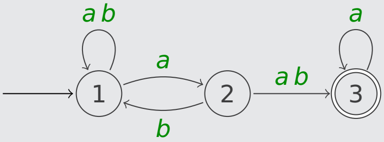
$B = \{1\}$

	a	b

	a	b



Example

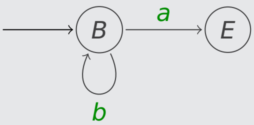


$$E = \{1, 2\}$$

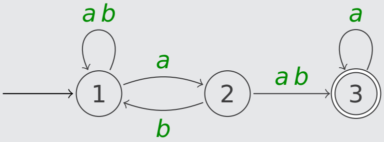
$$B = \{1\}$$

	a	b
B	E	B

	a	b



Example



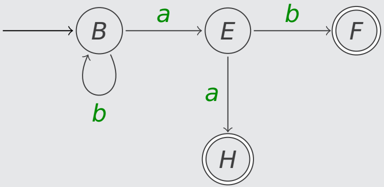
$$E = \{1,2\}$$

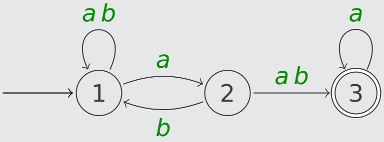
$$B = \{1\} \qquad F = \{1,3\}$$

$$H = \{1,2,3\}$$

	a	b
B	E	B

	a	b
E	H	F





$B = \{1\}$

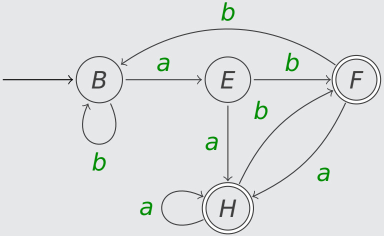
$E = \{1,2\}$

$F = \{1,3\}$

$H = \{1,2,3\}$

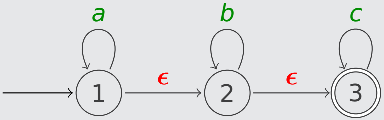
	a	b
B	E	B

	a	b
E	H	F
F	H	B
H	H	F

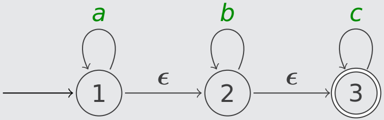


Outline

Example



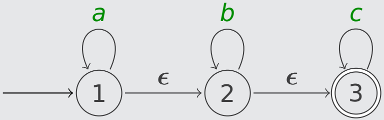
Example



Definitions

- NFA with ϵ–transitions (NFA$_\epsilon$) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that

Example



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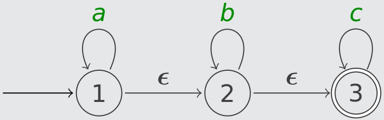
  1. $\epsilon \notin \Sigma$

Example



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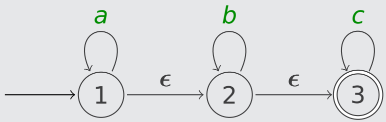

Definitions

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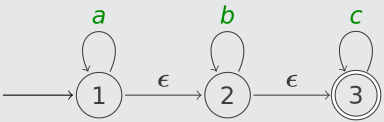

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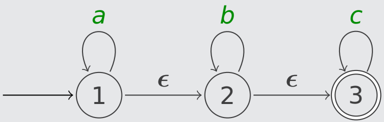

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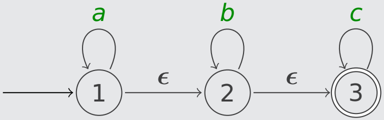

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Lemma

$C_\epsilon(A)$ is least extension of A that is closed under ϵ–transitions:

$$q \in C_\epsilon(A) \quad \Longrightarrow \quad \Delta_{N_\epsilon}(q, \epsilon) \subseteq C_\epsilon(A)$$

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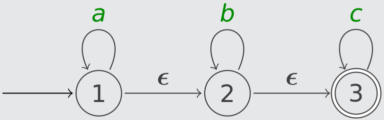

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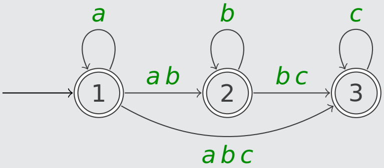
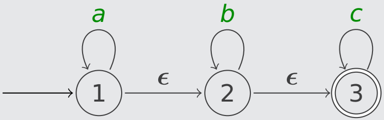


Example

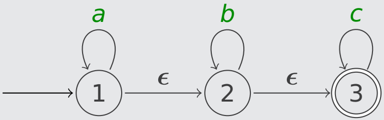
▶



Example

NFA$_\epsilon$ $N_1 = (\{1,2,3\}, \{a,b,c\}, \epsilon, \Delta_1, \{1\}, \{3\})$ with

Δ_1	a	b	c	ϵ
1	$\{1\}$	$\varnothing$	$\varnothing$	$\{2\}$
2	$\varnothing$	$\{2\}$	$\varnothing$	$\{3\}$
3	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$



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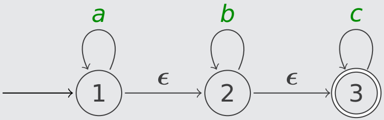

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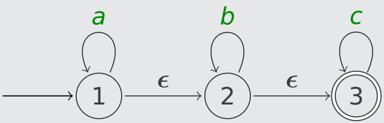

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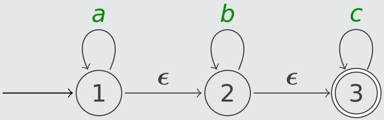

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1			
2			
3			

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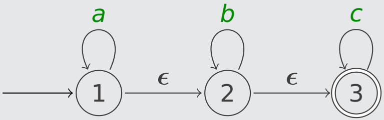

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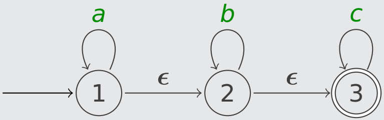

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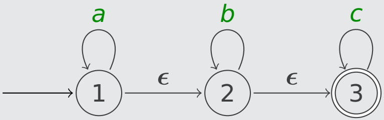

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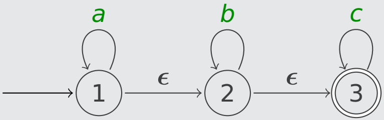

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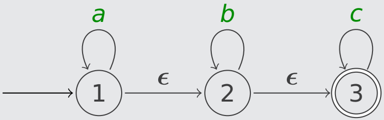

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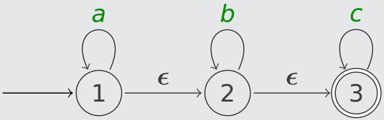

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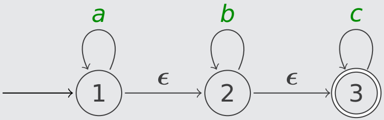

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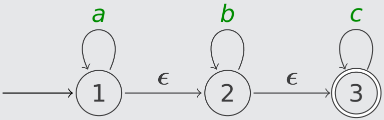

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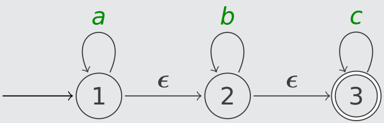

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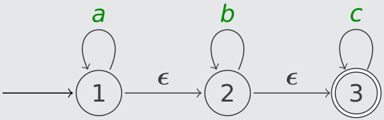

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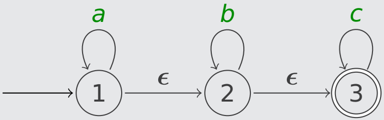
- $F_2 = \{q \mid C_\epsilon(\{q\}) \cap \{3\} \neq \varnothing\} = \{q \mid 3 \in C_\epsilon(\{q\})\} = \{1,2,3\}$
Δ_2	a	b	c
1	$\{1,2,3\}$	$\{2,3\}$	
2			
3			

$$\Delta_2(1,b) = \widehat{\Delta}_1(\{1\},b) = \bigcup \left\{ C_\epsilon(\Delta_1(q,b)) \;\middle|\; q \in \widehat{\Delta}_1(\{1\},\epsilon) \right\}$$
$$= C_\epsilon(\quad \varnothing \quad) \cup C_\epsilon(\quad \{2\} \quad) \cup C_\epsilon(\quad \varnothing \quad) = \{2,3\}$$

Example

NFA$_\epsilon$ $N_1 = (\{1,2,3\},\{a,b,c\},\epsilon,\Delta_1,\{1\},\{3\})$ with

Δ_1	a	b	c	ϵ
1	$\{1\}$	$\varnothing$	$\varnothing$	$\{2\}$
2	$\varnothing$	$\{2\}$	$\varnothing$	$\{3\}$
3	$\varnothing$	$\varnothing$	$\{3\}$	$\varnothing$



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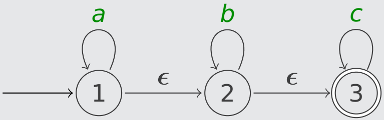

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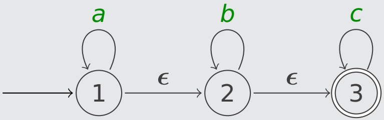

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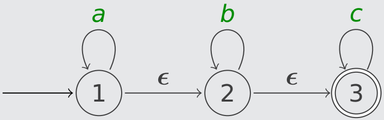

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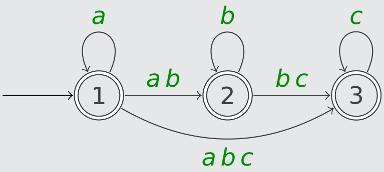

Outline

Question

What is the language accepted by the NFA_ϵ given by the following transition table ?

	ϵ	a	b
$\rightarrow$ 1	$\varnothing$	$\{2\}$	$\varnothing$
2	$\{3\}$	$\varnothing$	$\varnothing$
3	$\varnothing$	$\{4\}$	$\{2\}$
4 F	$\{1\}$	$\varnothing$	$\varnothing$

A $\{xyx \mid x \in \{a\} \text{ and } y \in \{b\}^*\}$

B the set of all strings over $\{a, b\}$ starting and ending with a

C $\{xyz \mid x, z \in \{a, b\} \text{ and } y \in \{aab\}^*\}$

D $\{\{a\}\{b\}^*\{a\}\}^+$

Outline

Theorem

regular sets are effectively closed under union, concatenation, and asterate

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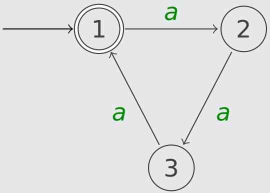
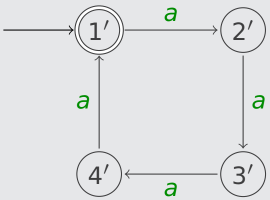
$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$



$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



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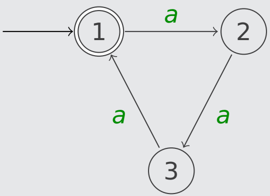

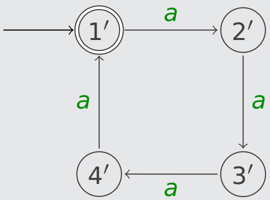

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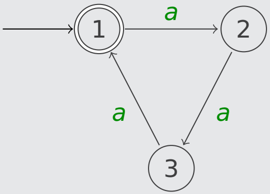

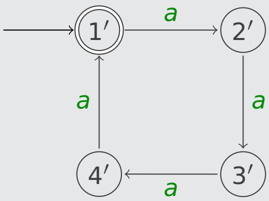

$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3 \text{ or } 4\}$$

Theorem

regular sets are effectively closed under union, concatenation, and asterate

Proof

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

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  4. $\Delta(q, a) = \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \\ \Delta_2(q, a) & \text{if } q \in Q_2 \end{cases}$

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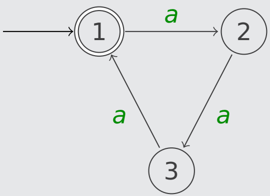

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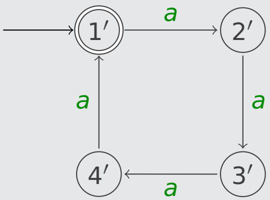
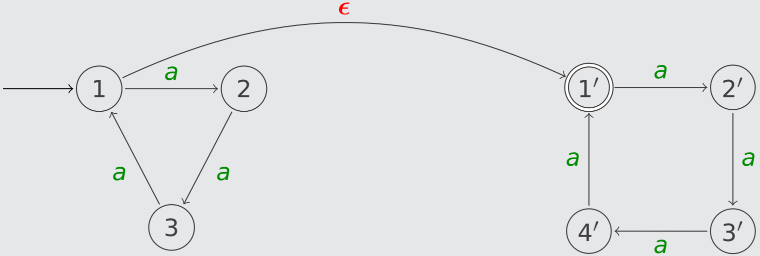


Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$ $\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



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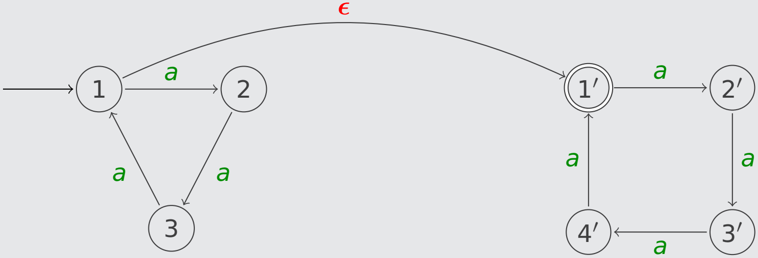

$\{x \in \{a\}^* \mid |x| \notin \{1, 2, 5\}\}$

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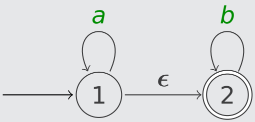
$$\{a\}^*\{b\}^*$$



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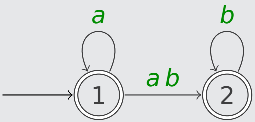
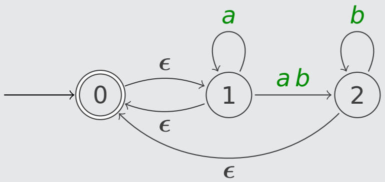


Example

$$(\{a\}^*\{b\}^*)^* = \{a,b\}^*$$



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Outline

Definitions

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  $\Delta_N((p,2),a) = \{(q,2) \mid \delta_M(p,a) = q\} \quad \text{for all } a \in \Sigma$

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② $$\begin{aligned}\Delta_N((p,0),a) &= \{(q,0) \mid \delta_M(p,a) = q\} \cup \{(q,1) \mid \delta_M(p,b) = q \text{ for some } b \neq a\}\\ \Delta_N((p,1),a) &= \{(q,1) \mid \delta_M(p,a) = q\} \cup \{(q,2) \mid \delta_M(p,b) = q \text{ for some } b \neq a\}\\ \Delta_N((p,2),a) &= \{(q,2) \mid \delta_M(p,a) = q\} \quad \text{for all } a \in \Sigma\end{aligned}$$

③ $S_N = \{(s_M, 0)\}$

Lemma

$A \subseteq \{0,1\}^*$ is regular $\implies$ $N_2(A)$ is regular

Proof

- $A = L(M)$ for DFA $M = (Q_M, \{0,1\}, \delta_M, s_M, F_M)$
- define NFA $N = (Q_N, \{0,1\}, \Delta_N, S_N, F_N)$ with

  ① $Q_N = Q_M \times \{0,1,2\}$

  ② $\Delta_N((p,0),a) = \{(q,0) \mid \delta_M(p,a) = q\} \cup \{(q,1) \mid \delta_M(p,b) = q \text{ for some } b \neq a\}$

  $\Delta_N((p,1),a) = \{(q,1) \mid \delta_M(p,a) = q\} \cup \{(q,2) \mid \delta_M(p,b) = q \text{ for some } b \neq a\}$

  $\Delta_N((p,2),a) = \{(q,2) \mid \delta_M(p,a) = q\} \quad \text{for all } a \in \Sigma$

  ③ $S_N = \{(s_M, 0)\}$

  ④ $F_N = F_M \times \{0,1,2\}$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad \Longleftrightarrow \quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

for all p, $q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad\Longleftrightarrow\quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

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- $N_2(A) = \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\}$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad\Longleftrightarrow\quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

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- $N_2(A) = \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\}$

  $= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\}$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad \Longleftrightarrow \quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

  for all p, $q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

- $$\begin{aligned} N_2(A) &= \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\} \\ &= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid H(y,x) = k \text{ and } \widehat{\delta_M}(s_M,x) = q \text{ for some } x \in A,\ k \in \{0,1,2\} \text{ and } q \in F_M\} \end{aligned}$$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad \Longleftrightarrow \quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

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Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \iff \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

  for all p, $q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

- $$\begin{aligned} N_2(A) &= \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\} \\ &= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid H(y,x) = k \text{ and } \widehat{\delta_M}(s_M,x) = q \text{ for some } x \in A,\ k \in \{0,1,2\} \text{ and } q \in F_M\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } q \in F_M \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } (q,k) \in F_N\} \end{aligned}$$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad\iff\quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

  for all p, $q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

- $$\begin{aligned} N_2(A) &= \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\} \\ &= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid H(y,x) = k \text{ and } \widehat{\delta_M}(s_M,x) = q \text{ for some } x \in A,\ k \in \{0,1,2\} \text{ and } q \in F_M\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } q \in F_M \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } (q,k) \in F_N\} \\ &= \{y \mid \widehat{\Delta}_N(\{(s_M,0)\},y) \cap F_N \neq \varnothing\} \end{aligned}$$

Proof (cont'd)

- key property:

$$(q,j) \in \widehat{\Delta}_N(\{(p,i)\},y) \quad\Longleftrightarrow\quad \widehat{\delta_M}(p,x) = q \text{ for some } x \in \{0,1\}^*$$

  for all p, $q \in Q_M$, $y \in \{0,1\}^*$, $i, j \in \{0,1,2\}$ such that $|x| = |y|$ and $H(x,y) = j - i$

- $$\begin{aligned} N_2(A) &= \{y \mid H(y,x) \leqslant 2 \text{ for some } x \in A\} \\ &= \{y \mid H(y,x) = k \text{ for some } x \in A \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid H(y,x) = k \text{ and } \widehat{\delta_M}(s_M,x) = q \text{ for some } x \in A,\ k \in \{0,1,2\} \text{ and } q \in F_M\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } q \in F_M \text{ and } k \in \{0,1,2\}\} \\ &= \{y \mid (q,k) \in \widehat{\Delta}_N(\{(s_M,0)\},y) \text{ for some } (q,k) \in F_N\} \\ &= \{y \mid \widehat{\Delta}_N(\{(s_M,0)\},y) \cap F_N \neq \varnothing\} \\ &= L(N) \end{aligned}$$

Outline

1. Summary of Previous Lecture
2. Nondeterministic Finite Automata
3. Epsilon Transitions
4. Intermezzo
5. Closure Properties
6. Hamming Distance
7. **Further Reading**

Kozen

- Lecture 5 and 6

Kozen

- Lecture 5 and 6

Important Concepts

- ϵ–transition
- ϵ–closure
- asterate
- Hamming distance
- NFA
- NFA_ϵ
- subset construction

Kozen

- Lecture 5 and 6

Important Concepts

- ϵ–transition
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homework for October 24