Automata and Logic

Aart Middeldorp and Samuel Frontull

Outline

Definitions

- nondeterministic finite automaton (NFA) is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with

  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\Delta \colon Q \times \Sigma \to 2^Q$: transition function
  4. $S \subseteq Q$: set of start states
  5. $F \subseteq Q$: final (accept) states

- $\widehat{\Delta} \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}(A, \epsilon) = A \qquad\qquad \widehat{\Delta}(A, xa) = \bigcup_{q \in \widehat{\Delta}(A,x)} \Delta(q, a)$$

- $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

Theorem

every set accepted by NFA is regular

Definitions

- NFA with ϵ–transitions (NFA$_\epsilon$) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that
  1. $\epsilon \notin \Sigma$
  2. $M_\epsilon = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$
- ϵ–closure of set $A \subseteq Q$ is defined as $C_\epsilon(A) = \bigcup \{\widehat{\Delta}_{N_\epsilon}(A, x) \mid x \in \{\epsilon\}^*\}$
- $\widehat{\Delta}_N \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}_N(A, \epsilon) = C_\epsilon(A) \qquad \widehat{\Delta}_N(A, xa) = \bigcup \{C_\epsilon(\Delta(q, a)) \mid q \in \widehat{\Delta}_N(A, x)\}$$

Theorem

- every set accepted by NFA$_\epsilon$ is regular
- regular sets are effectively closed under union, concatenation, and asterate

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second−order logic
- Presburger arithmetic
- linear−time temporal logic

Outline

1. Summary of Previous Lecture
2. **Regular Expressions**
3. Intermezzo
4. Homomorphisms
5. Decision Problems
6. Further Reading

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- regular expression α over alphabet Σ:

$$a \in \Sigma \qquad \epsilon \qquad \varnothing \qquad \beta + \gamma \qquad \beta\gamma \qquad \beta^*$$

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regular expressions α and β are equivalent $(\alpha \equiv \beta)$ if $L(\alpha) = L(\beta)$

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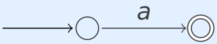
$a \in \Sigma$	$\{a\}$	
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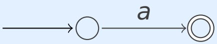
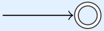
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
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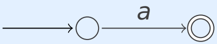
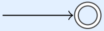
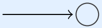
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
$\varnothing$	$\varnothing$	

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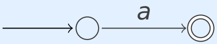
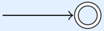
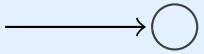
$a \in \Sigma$	$\{a\}$	
ϵ	$\{\epsilon\}$	
$\varnothing$	$\varnothing$	

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$L(\beta)$ and $L(\gamma)$ are regular according to induction hypothesis

$\Longrightarrow \quad L(\alpha)$ is regular according to closure properties of regular sets

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Theorem

finite automata and regular expressions are equivalent:

$$\text{for all } A \subseteq \Sigma^* \quad A \text{ is regular} \quad \Longleftrightarrow \quad A = L(\alpha) \text{ for some regular expression } \alpha$$

Proof ($\Longrightarrow$)

given NFA $N = (Q, \Sigma, \Delta, S, F)$ with $Q = \{1, \dots, n\}$ and $S = \{1\}$

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- define system of equations

$$X_i = \left(\bigcup_{a \in \Sigma} \bigcup_{j \in \Delta(i,a)} \{a\} X_j \right) \cup \begin{cases} \{\epsilon\} & \text{if } i \in F \\ \varnothing & \text{otherwise} \end{cases}$$

  with unknowns $X_1, \dots, X_n$

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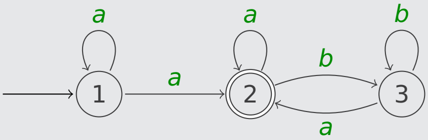
- transform X_1 into regular expression by successive substitution and Arden's lemma

Example



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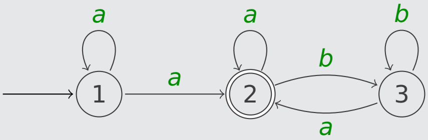

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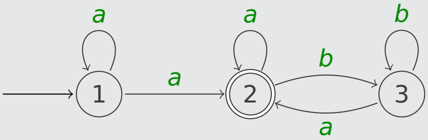

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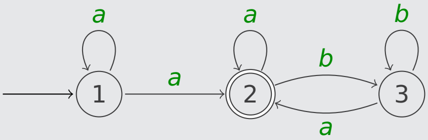
$$X_1 = a^* a X_2 \qquad X_2 = a^*(b X_3 + \epsilon) \qquad X_3 = b^* a X_2 \qquad \text{(Arden's lemma)}$$

Example



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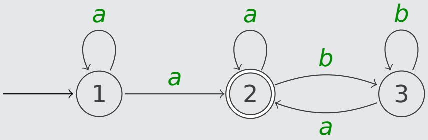
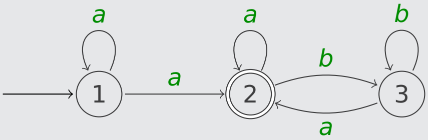


$$
\begin{array}{lll}
X_1 = aX_1 + aX_2 & X_2 = aX_2 + bX_3 + \epsilon & X_3 = aX_2 + bX_3 \\
X_1 = a^*aX_2 & X_2 = a^*(bX_3 + \epsilon) & X_3 = b^*aX_2 \quad \text{(Arden's lemma)} \\
X_1 = a^*aX_2 & X_2 = a^*(bb^*aX_2 + \epsilon) & \text{(substitute)} \\
X_1 = a^*aX_2 & X_2 = a^*bb^*aX_2 + a^* & \text{(distribute)}
\end{array}
$$

Example



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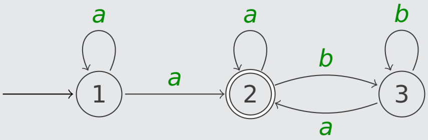
Example



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\end{array}
$$

Outline

Question

Which of the following strings belong to $L((a + ba + bab)^*)$?

A ϵ

B $ababa$

C all strings over $\{a, b\}$ that start with a

D all strings over $\{a, b\}$ that do not contain two consecutive b's

Outline

Theorem

regular sets are effectively closed under homomorphic image and preimage

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regular sets are effectively closed under homomorphic image and preimage

Definitions

- homomorphism is mapping $h\colon \Sigma^* \to \Gamma^*$ such that

$$h(\epsilon) = \epsilon \qquad\qquad h(xy) = h(x)h(y)$$

Theorem

regular sets are effectively closed under homomorphic image and preimage

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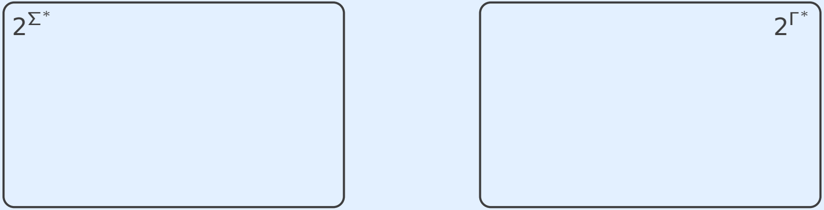
- if $B \subseteq \Gamma^*$ then $h^{-1}(B) = \{x \mid h(x) \in B\} \subseteq \Sigma^*$ "preimage of B under h"



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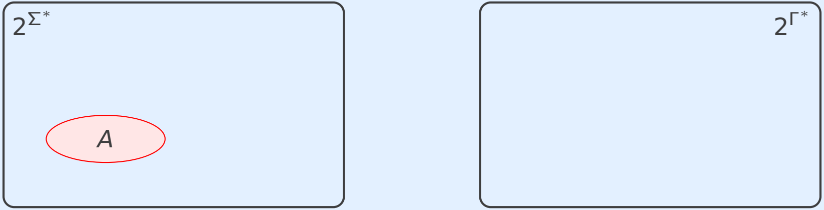

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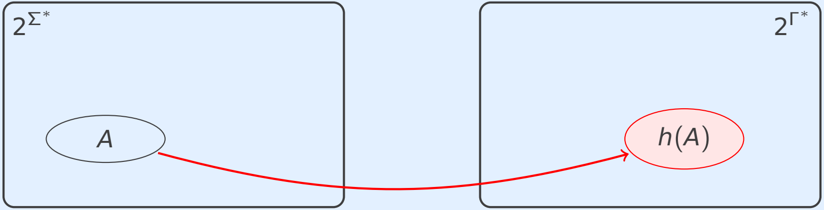

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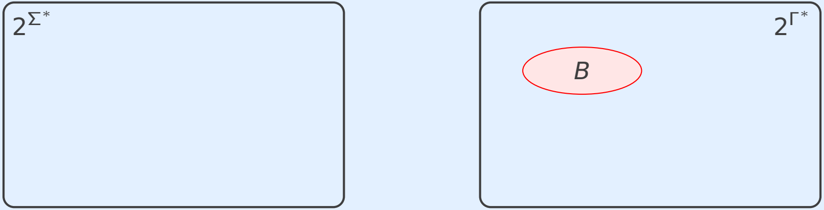

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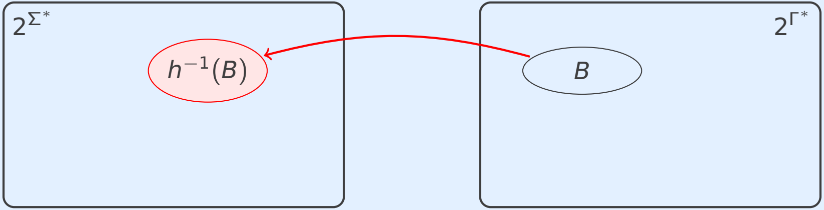

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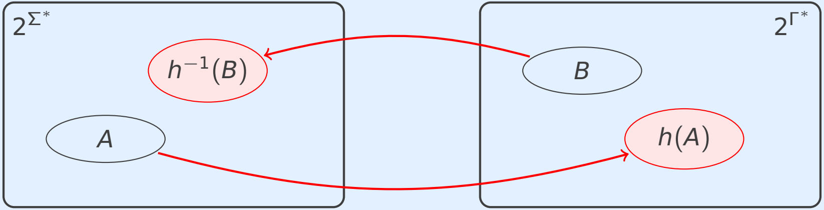
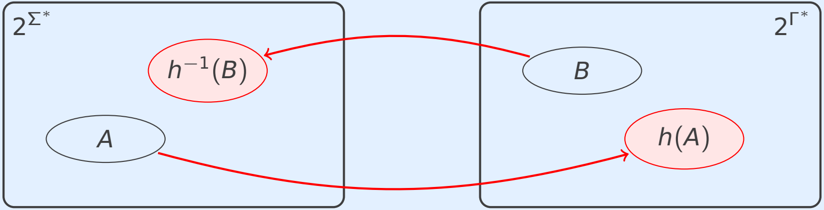


- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h^{-1}(h(A)) \supseteq A$
- $h(h^{-1}(B)) \subseteq B$



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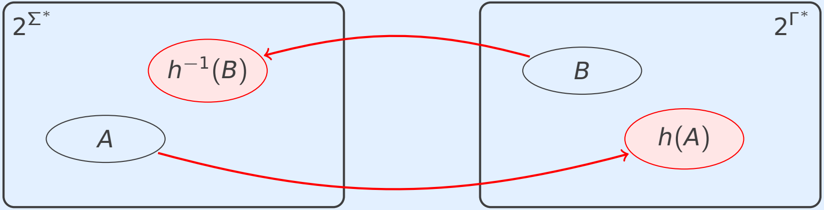
Example

$\Sigma = \Gamma = \{0,1\} \quad h(0) = 11 \quad h(1) = 1$



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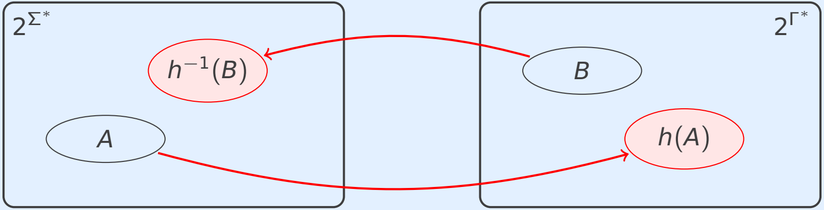
- $h^{-1}(h(A)) = h^{-1}(\{11\}) = \{0, 11\} \supsetneq A$



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- $h(h^{-1}(B)) = h(\varnothing) = \varnothing \subsetneq B$

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$A \subseteq \{0,1\}^*$ is regular $\implies$ $\{xy \mid x1y \in A\}$ is regular

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- $h^{-1}(A) \cap L((0+1)^*2(0+1)^*) = \{x2y \mid x1y \in A\}$

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Theorem

regular sets are effectively closed under homomorphic image and preimage

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Proof

- NFA $M = (Q, \Gamma, \Delta, S, F)$
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- NFA $M = (Q, \Gamma, \Delta, S, F)$
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- $h^{-1}(L(M)) = L(M')$ for NFA $M' = (Q, \Sigma, \Delta', S, F)$ with $\Delta'(q, a) = \widehat{\Delta}(\{q\}, h(a))$

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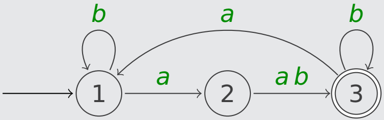
- NFA $M = (Q, \Gamma, \Delta, S, F)$
- homomorphism $h \colon \Sigma^* \to \Gamma^*$
- $h^{-1}(L(M)) = L(M')$ for NFA $M' = (Q, \Sigma, \Delta', S, F)$ with $\Delta'(q, a) = \widehat{\Delta}(\{q\}, h(a))$
- claim: $\widehat{\Delta'}(A, x) = \widehat{\Delta}(A, h(x))$ for all $A \subseteq Q$ and $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$

Example

- DFA M



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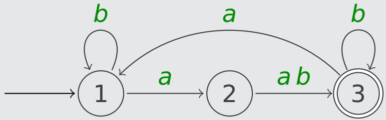

- homomorphism $h\colon \{a, b, c\}^* \to \{a, b\}^*$

$$h(a) = aa \qquad h(b) = \epsilon \qquad h(c) = bab$$

Example

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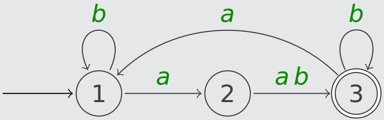

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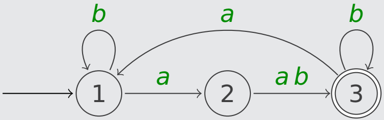

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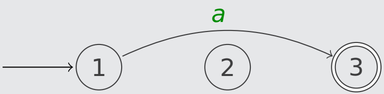

$$\delta'(1,a) = \widehat{\delta}(1,aa) = 3$$

Example

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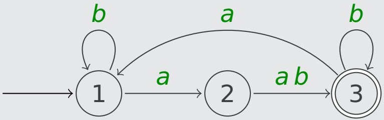

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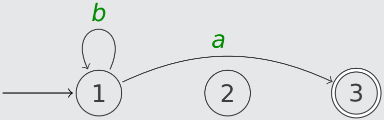

$$\delta'(1, b) = \widehat{\delta}(1, \epsilon) = 1$$

Example

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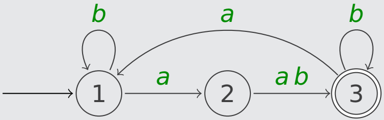

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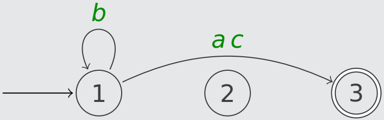

$$\delta'(1,c) = \widehat{\delta}(1,bab) = 3$$

Example

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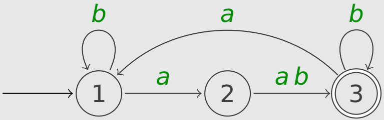

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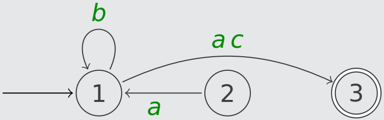
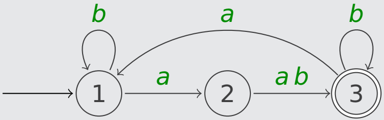


$$\delta'(2,a) = \widehat{\delta}(2,aa) = 1$$

Example

- DFA M



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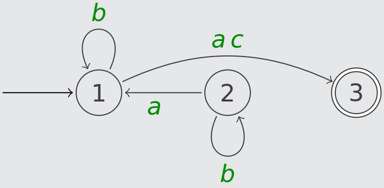
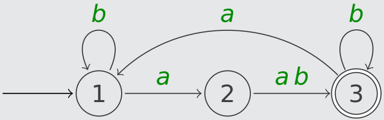


$$\delta'(2,b) = \widehat{\delta}(2,\epsilon) = 2$$

Example

- DFA M



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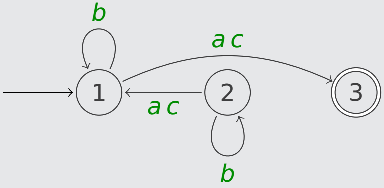

$$\delta'(2,c) = \widehat{\delta}(2,bab) = 1$$

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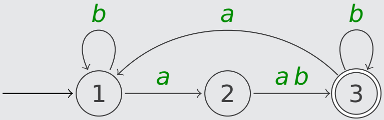
- DFA M



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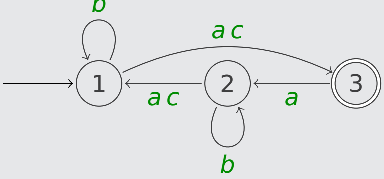

$$\delta'(3,a) = \widehat{\delta}(3,aa) = 2$$

Example

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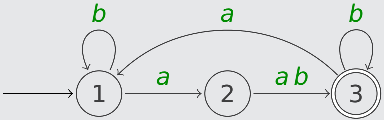

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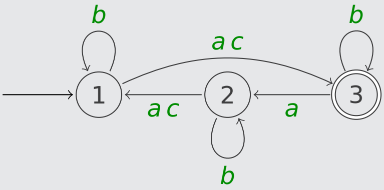
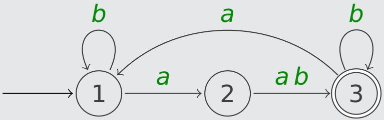


$\delta'(3,b) = \widehat{\delta}(3,\epsilon) = 3$

Example

- DFA M



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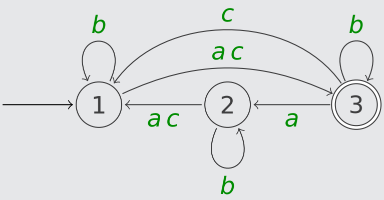

$$\delta'(3,c) = \widehat{\delta}(3,bab) = 1$$

Theorem

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Proof

- regular expression α over Σ
- homomorphism $h \colon \Sigma^* \to \Gamma^*$

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Proof

- regular expression α over Σ
- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h(L(\alpha)) = L(\alpha')$ for regular expression α' defined inductively:

$$
\begin{aligned}
a' &= h(a) \quad \text{for } a \in \Sigma & \qquad (\beta + \gamma)' &= \beta' + \gamma' \\
\epsilon' &= \epsilon & (\beta\gamma)' &= \beta'\gamma' \\
\varnothing' &= \varnothing & \beta^{*\prime} &= \beta'^{*}
\end{aligned}
$$

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- Hamming distance $H(x,y)$ is number of places where bit strings x and y differ
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representation of regular sets (DFA, NFA, regular expression) may affect complexity of decision problems

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Kozen

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Important Concepts

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- homomorphism
- homomorphic image
- homomorphic preimage
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homework for October 24