



Automata and Logic

Aart Middeldorp and Samuel Frontull

Outline

- 1. Summary of Previous Lecture**
- 2. WMSO Definability**
- 3. Intermezzo**
- 4. WMSO Definability**
- 5. MONA**
- 6. Further Reading**

Definition

equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

$$x \equiv_M y \iff \hat{\delta}(s, x) = \hat{\delta}(s, y)$$

Lemmata

- ▶ $\equiv_M$ is **right congruent**: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ for all $a \in \Sigma$ $xa \equiv_M ya$
- ▶ $\equiv_M$ **refines** $L(M)$: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ either $x, y \in L(M)$ or $x, y \notin L(M)$
- ▶ $\equiv_M$ is of **finite index**: $\equiv_M$ has finitely many equivalence classes

Definition

Myhill–Nerode relation for $L \subseteq \Sigma^*$ is right congruent equivalence relation of finite index on Σ^* that refines L

Definition

for any set $L \subseteq \Sigma^*$ equivalence relation $\equiv_L$ on Σ^* is defined as follows:

$$x \equiv_L y \iff \text{for all } z \in \Sigma^* \quad (xz \in L \iff yz \in L)$$

Theorem

① for every regular set $L \subseteq \Sigma^*$

there exists **one-to-one correspondence** (up to isomorphism of automata) between

- ▶ DFAs for L with input alphabet Σ and without inaccessible states
- ▶ Myhill–Nerode relations for L

② for every set $L \subseteq \Sigma^*$

L is regular $\iff L$ admits Myhill–Nerode relation $\iff \equiv_L$ is of finite index

③ regular sets are WMSO definable

Automata

- ▶ (deterministic, nondeterministic, alternating) **finite automata**
- ▶ regular expressions
- ▶ (alternating) Büchi automata

Logic

- ▶ **(weak) monadic second-order logic**
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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Definitions

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Example

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$$FV(\varphi) = (x, Y)$$

Definitions

- ▶ $FV(\varphi)$ denotes list of free variables in φ in fixed order with first-order variables preceding second-order ones
- ▶ assignment for φ with $FV(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is tuple $(i_1, \dots, i_m, l_1, \dots, l_n)$ such that $i_1, \dots, i_m$ are elements of $\mathbb{N}$ and $l_1, \dots, l_n$ are finite subsets of $\mathbb{N}$

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Example

$$\varphi = \exists X. X(x) \rightarrow \exists y. x < y \wedge Y(y)$$

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Notation

$$(i, l) \text{ for } (i_1, \dots, i_m, l_1, \dots, l_n)$$

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example

x_1	0	0	0	1	0	0
X_1	1	0	1	0	1	0
X_2	0	0	0	0	0	0

Example

	0	1	2	3	4	5
x_1	0	0	0	1	0	0
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induced assignment

$$i_1 = 3$$

Example

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induced assignment

$$i_1 = 3$$

$$I_1 = \{0, 2, 4\}$$

Example

	0	1	2	3	4	5	induced assignment
x_1	0	0	0	1	0	0	$i_1 = 3$
X_1	1	0	1	0	1	0	$I_1 = \{0, 2, 4\}$
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Example

	0	1	2	3	4	5	induced assignment
x_1	0	0	0	1	0	0	$i_1 = 3$
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X_2	0	0	0	0	0	0	$I_2 = \emptyset$

Remark

assignments are identified with strings over $\{0, 1\}^{m+n}$ (here $k = m + n$)

$$\begin{array}{cccc}
 a_{11} & a_{21} & \cdots & a_{\ell 1} \\
 \vdots & \vdots & & \vdots \\
 a_{1k} & a_{2k} & \cdots & a_{\ell k}
 \end{array}
 \approx
 \begin{pmatrix} a_{11} \\ \vdots \\ a_{1k} \end{pmatrix}
 \begin{pmatrix} a_{21} \\ \vdots \\ a_{2k} \end{pmatrix}
 \cdots
 \begin{pmatrix} a_{\ell 1} \\ \vdots \\ a_{\ell k} \end{pmatrix}$$

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string over $\{0, 1\}^{m+n}$ is **m -admissible** if first m rows contain exactly one 1 each

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if x is m -admissible then $x\mathbf{0}$ is m -admissible and $\underline{x} = \underline{x\mathbf{0}}$

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- ▶ if $x, y \in (\{0, 1\}^{m+n})^*$ induce same assignment then $x = y\mathbf{0} \cdots \mathbf{0}$ or $y = x\mathbf{0} \cdots \mathbf{0}$

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- ▶ $\epsilon \in (\{0, 1\}^{m+n})^*$ is m -admissible if and only if $m = 0$

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- ▶ $\epsilon \in (\{0, 1\}^{m+n})^*$ is m -admissible if and only if $m = 0$
- ▶ if $\underline{x} = (i, I)$ then $|x| > k$ for all $k \in \{i_1, \dots, i_m\} \cup I_1 \cup \dots \cup I_n$

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Lemma

set of m -admissible strings over $\{0, 1\}^{m+n}$ is regular

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Question

Consider the following three strings over $\{0, 1\}^{2+2}$. Which statements hold ?

	0	1	2		0	1	2		0	1	2	3
x_1	0	1	0		0	1	0		0	1	0	0
x_2	1	0	1		0	0	1		0	0	1	0
X_1	1	0	0		1	0	0		1	0	0	0
X_2	0	1	1		0	1	1		0	1	1	0

- A** the first string is 2-admissible
- B** the second and third string induce the same assignment
- C** $X_2(x_1)$ is satisfied by the first string's induced assignment
- D** the third string induces the assignment $i_1 = 1, i_2 = 2, I_1 = \{0\}, I_2 = \{1, 2\}$



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Proof (construction)

define DFA $\mathcal{A}_{m,n} = (Q, \Sigma, \delta, s, F)$

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- ① $\Sigma = \{0, 1\}^{m+n}$
- ② $Q = 2^{\{1, \dots, m\}} \cup \{\perp\}$

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② $Q = 2^{\{1, \dots, m\}} \cup \{\perp\}$

③ $s = \{1, \dots, m\}$

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⑤ $\delta(q, a) = \begin{cases} q - I & \text{if } q \subseteq \{1, \dots, m\} \text{ and } I \subseteq q \\ \perp & \text{if } q \subseteq \{1, \dots, m\} \text{ and } I \not\subseteq q \\ \perp & \text{if } q = \perp \end{cases}$

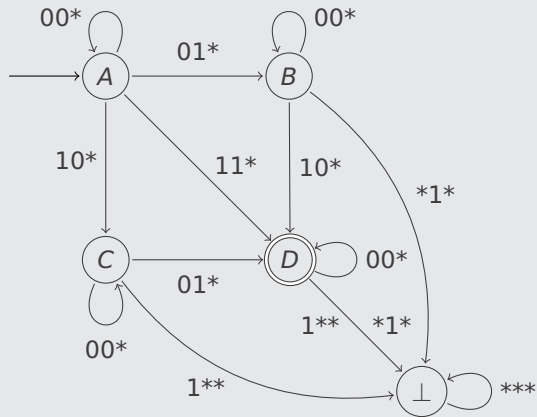
with $I = \{i \in \{1, \dots, m\} \mid a_i = 1\}$

Example

DFA $\mathcal{A}_{2,1}$

Example

DFA $\mathcal{A}_{2,1}$ with $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$, $D = \emptyset$



Definition

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Example

- ▶ $\varphi(x, X) = \forall y. y < x \rightarrow X(y)$
- ▶ $L_a(\varphi) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^* \left[\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \left[\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]^*$

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Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

Proof

induction on φ

induction on φ

► $\varphi = \perp$

► $\varphi = x < y$

► $\varphi = X(x)$

► $\varphi = \neg\psi$

► $\varphi = \varphi_1 \vee \varphi_2$

Proof

induction on φ

► $\varphi = \perp \quad \implies \quad L_a(\varphi) = \emptyset$

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- ▶ $\varphi = \perp \implies L_a(\varphi) = \emptyset$
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- ▶ $\varphi = \perp \implies L_a(\varphi) = \emptyset$
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- ▶ $\varphi = \neg\psi \implies L_a(\psi) \text{ is regular} \implies \sim L_a(\psi) \text{ is regular}$
 $\implies L_a(\varphi) = \sim L_a(\psi) \cap L(\mathcal{A}_{m,n}) \text{ is regular (for suitable } m \text{ and } n)$
- ▶ $\varphi = \varphi_1 \vee \varphi_2$

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- ▶ $\varphi = \varphi_1 \vee \varphi_2$
 $L_a(\varphi_1) \text{ and } L_a(\varphi_2) \text{ are regular}$

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$L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

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- ▶ $\varphi = \neg\psi \implies L_a(\psi) \text{ is regular} \implies \sim L_a(\psi) \text{ is regular}$
 $\implies L_a(\varphi) = \sim L_a(\psi) \cap L(\mathcal{A}_{m,n}) \text{ is regular (for suitable } m \text{ and } n)$
- ▶ $\varphi = \varphi_1 \vee \varphi_2$ with $\text{FV}(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$

$L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

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Definition

homomorphism

$$\text{drop}_i: (\{0,1\}^k)^* \rightarrow (\{0,1\}^{k-1})^*$$

is defined for $1 \leq i \leq k$ by dropping i -th component from vectors in $\{0,1\}^k$

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► $A \subseteq (\{0,1\}^k)^*$ is regular $\implies \text{drop}_i(A) \subseteq (\{0,1\}^{k-1})^*$ is regular

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Lemmata

- ▶ $A \subseteq (\{0,1\}^k)^*$ is regular $\implies \text{drop}_i(A) \subseteq (\{0,1\}^{k-1})^*$ is regular
- ▶ $B \subseteq (\{0,1\}^{k-1})^*$ is regular $\implies \text{drop}_i^{-1}(B) \subseteq (\{0,1\}^k)^*$ is regular

Example

$\varphi = x < y \vee X(x)$ with $FV(\varphi) = (x, y, X)$

- ▶ $L_a(x < y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^*$
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- ▶ $L_1 = \text{drop}_3^{-1} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \right)$

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$$\triangleright L_2 = \text{drop}_2^{-1} \left(\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^* \right)$$

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$\varphi = x < y \vee X(x)$ with $FV(\varphi) = (x, y, X)$

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$$\triangleright L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{2,1})$$

induction on φ

- ▶ $\varphi = \perp \implies L_a(\varphi) = \emptyset$
- ▶ $\varphi = x < y \implies L_a(\varphi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^*$ or $L_a(\varphi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^*$
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- ▶ $\varphi = \neg\psi \implies L_a(\psi) \text{ is regular} \implies \sim L_a(\psi) \text{ is regular}$
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- ▶ $\varphi = \varphi_1 \vee \varphi_2$ with $FV(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$

$L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

applications of inverse homomorphism drop_i^{-1} to $L_a(\varphi_1)$ and $L_a(\varphi_2)$ yield regular sets $L_1, L_2 \subseteq (\{0, 1\}^{m+n})^*$ such that $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{m,n})$

Proof (cont'd)

► $\varphi = \exists x. \psi$

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Proof (cont'd)

► $\varphi = \exists x. \psi \implies L_a(\psi)$ is regular

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- ▶ $\varphi = \exists x. \psi \implies L_a(\psi)$ is regular
 $\implies \text{drop}_i(L_a(\psi))$ is regular where i is position of x in $\text{FV}(\psi)$
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Example

$\varphi = \exists X. \psi$ with $\psi = X(x) \wedge \exists y. x < y \wedge X(y)$

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- ▶ $\text{drop}_2(L_a(\psi)) = (0 + 0)^* 1 (0 + 0)^* 0 (0 + 0)^*$

Proof (cont'd)

- ▶ $\varphi = \exists x. \psi \implies L_a(\psi)$ is regular
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- ▶ $\text{drop}_2(L_a(\psi)) = (0 + 0)^* 1 (0 + 0)^* 0 (0 + 0)^* = 0^* 1 0 0^* \neq 0^* 1 0^* = L_a(\varphi)$

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Definition

$$\text{stz}(A) = \{x \mid x\mathbf{0} \cdots \mathbf{0} \in A\} \text{ for } A \subseteq (\{0,1\}^{m+n})^*$$

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► DFA $M = (Q, \Sigma, \delta, s, F)$ with $L(M) = A$

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- ▶ $L(M') = \text{stz}(A)$

Proof (cont'd)

- ▶ $\varphi = \exists x. \psi \implies L_a(\psi)$ is regular
 - $\implies \text{drop}_i(L_a(\psi))$ is regular where i is position of x in $\text{FV}(\psi)$
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Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$

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Procedure

- ① eliminate assignments which do not correspond to string in Σ^*

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Lemma

$\{0^k 10^l \mid k + 1 + l = |\Sigma|\}^*$ is regular

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- ① eliminate assignments which do not correspond to string in Σ^*

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$L_a(\varphi) \cap \{0^k 10^l \mid k + 1 + l = |\Sigma|\}^*$ is regular

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$ using regularity preserving operations

Procedure

- ① eliminate assignments which do not correspond to string in Σ^*
- ② map strings in 0^*10^* to elements of Σ

Lemma

$L_a(\varphi) \cap \{0^k10^l \mid k + 1 + l = |\Sigma|\}^*$ is regular

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$ using regularity preserving operations

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Lemma

$L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*$ is regular

Final Task

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Corollary

WMSO definable sets are regular

Outline

1. Summary of Previous Lecture
2. WMSO Definability
3. Intermezzo
4. WMSO Definability
- 5. MONA**
6. Further Reading

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Demo

<https://www.brics.dk/mona/>

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- ▶ Section 10.9 of **Einführung in die mathematische Logik** (Springer Spektrum 2018)

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- ▶ Section 3 of **MONA Version 1.4 User Manual** (2001)

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Important Concepts

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homework for November 14