



Automata and Logic

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Outline

- 1. Summary of Previous Lecture**
- 2. WMSO Definability**
- 3. Intermezzo**
- 4. WMSO Definability**
- 5. MONA**
- 6. Further Reading**

Definition

equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

$$x \equiv_M y \iff \hat{\delta}(s, x) = \hat{\delta}(s, y)$$

Lemmata

- ▶ $\equiv_M$ is **right congruent**: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ for all $a \in \Sigma$ $xa \equiv_M ya$
- ▶ $\equiv_M$ **refines** $L(M)$: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ either $x, y \in L(M)$ or $x, y \notin L(M)$
- ▶ $\equiv_M$ is of **finite index**: $\equiv_M$ has finitely many equivalence classes

Definition

Myhill–Nerode relation for $L \subseteq \Sigma^*$ is right congruent equivalence relation of finite index on Σ^* that refines L

Definition

for any set $L \subseteq \Sigma^*$ equivalence relation $\equiv_L$ on Σ^* is defined as follows:

$$x \equiv_L y \iff \text{for all } z \in \Sigma^* \quad (xz \in L \iff yz \in L)$$

Theorem

① for every regular set $L \subseteq \Sigma^*$

there exists **one-to-one correspondence** (up to isomorphism of automata) between

- ▶ DFAs for L with input alphabet Σ and without inaccessible states
- ▶ Myhill–Nerode relations for L

② for every set $L \subseteq \Sigma^*$

$$L \text{ is regular} \iff L \text{ admits Myhill–Nerode relation} \iff \equiv_L \text{ is of finite index}$$

③ regular sets are WMSO definable

Automata

- ▶ (deterministic, nondeterministic, alternating) **finite automata**
- ▶ regular expressions
- ▶ (alternating) Büchi automata

Logic

- ▶ **(weak) monadic second-order logic**
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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Definitions

- ▶ $FV(\varphi)$ denotes list of free variables in φ in fixed order with first-order variables preceding second-order ones
- ▶ assignment for φ with $FV(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is tuple $(i_1, \dots, i_m, l_1, \dots, l_n)$ such that $i_1, \dots, i_m$ are elements of $\mathbb{N}$ and $l_1, \dots, l_n$ are finite subsets of $\mathbb{N}$

Example

$$\varphi = \exists X. X(x) \rightarrow \exists y. x < y \wedge Y(y)$$

$$FV(\varphi) = (x, Y)$$

Notation

$$(i, l) \text{ for } (i_1, \dots, i_m, l_1, \dots, l_n)$$

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example

	0	1	2	3	4	5	induced assignment
x_1	0	0	0	1	0	0	$i_1 = 3$
X_1	1	0	1	0	1	0	$I_1 = \{0, 2, 4\}$
X_2	0	0	0	0	0	0	$I_2 = \emptyset$

Remark

assignments are identified with strings over $\{0, 1\}^{m+n}$ (here $k = m + n$)

$$\begin{array}{cccc} a_{11} & a_{21} & \cdots & a_{\ell 1} \\ \vdots & \vdots & & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{\ell k} \end{array} \approx \begin{pmatrix} a_{11} \\ \vdots \\ a_{1k} \end{pmatrix} \begin{pmatrix} a_{21} \\ \vdots \\ a_{2k} \end{pmatrix} \cdots \begin{pmatrix} a_{\ell 1} \\ \vdots \\ a_{\ell k} \end{pmatrix}$$

Definition

string over $\{0, 1\}^{m+n}$ is **m -admissible** if first m rows contain exactly one 1 each

Remarks

- ▶ every m -admissible string x induces assignment $\underline{x}$
- ▶ every assignment is induced by (**not necessarily unique**) m -admissible string:
if x is m -admissible then $x\mathbf{0}$ is m -admissible and $\underline{x} = \underline{x\mathbf{0}}$
- ▶ if $x, y \in (\{0, 1\}^{m+n})^*$ induce same assignment then $x = y\mathbf{0} \cdots \mathbf{0}$ or $y = x\mathbf{0} \cdots \mathbf{0}$
- ▶ $\epsilon \in (\{0, 1\}^{m+n})^*$ is m -admissible if and only if $m = 0$
- ▶ if $\underline{x} = (i, I)$ then $|x| > k$ for all $k \in \{i_1, \dots, i_m\} \cup I_1 \cup \dots \cup I_n$

Lemma

set of m -admissible strings over $\{0, 1\}^{m+n}$ is regular

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Question

Consider the following three strings over $\{0, 1\}^{2+2}$. Which statements hold ?

	0	1	2		0	1	2		0	1	2	3
x_1	0	1	0		0	1	0		0	1	0	0
x_2	1	0	1		0	0	1		0	0	1	0
X_1	1	0	0		1	0	0		1	0	0	0
X_2	0	1	1		0	1	1		0	1	1	0

- A** the first string is 2-admissible
- B** the second and third string induce the same assignment
- C** $X_2(x_1)$ is satisfied by the first string's induced assignment
- D** the third string induces the assignment $i_1 = 1, i_2 = 2, l_1 = \{0\}, l_2 = \{1, 2\}$



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Lemma

set of m -admissible strings over $\{0, 1\}^{m+n}$ is regular

Proof (construction)

define DFA $\mathcal{A}_{m,n} = (Q, \Sigma, \delta, s, F)$ with

① $\Sigma = \{0, 1\}^{m+n}$

② $Q = 2^{\{1, \dots, m\}} \cup \{\perp\}$

③ $s = \{1, \dots, m\}$

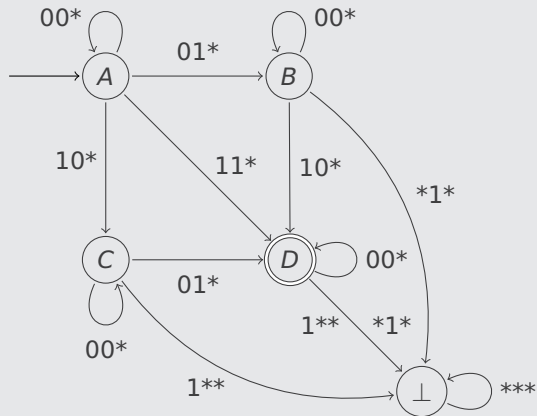
④ $F = \{\emptyset\}$

⑤ $\delta(q, a) = \begin{cases} q - I & \text{if } q \subseteq \{1, \dots, m\} \text{ and } I \subseteq q \\ \perp & \text{if } q \subseteq \{1, \dots, m\} \text{ and } I \not\subseteq q \\ \perp & \text{if } q = \perp \end{cases}$

with $I = \{i \in \{1, \dots, m\} \mid a_i = 1\}$

Example

DFA $\mathcal{A}_{2,1}$ with $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$, $D = \emptyset$



Definition

$$L_a(\varphi) = \{x \in (\{0, 1\}^{m+n})^* \mid x \text{ is } m\text{-admissible and } \underline{x} \models \varphi\}$$

Example

- ▶ $\varphi(x, X) = \forall y. y < x \rightarrow X(y)$
- ▶ $L_a(\varphi) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^* \left[\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\right] \left[\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right]^*$

Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

induction on φ

- ▶ $\varphi = \perp \implies L_a(\varphi) = \emptyset$
- ▶ $\varphi = x < y \implies L_a(\varphi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^*$ or $L_a(\varphi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^*$
- ▶ $\varphi = X(x) \implies L_a(\varphi) = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^*$
- ▶ $\varphi = \neg\psi \implies L_a(\psi) \text{ is regular} \implies \sim L_a(\psi) \text{ is regular}$
 $\implies L_a(\varphi) = \sim L_a(\psi) \cap L(\mathcal{A}_{m,n}) \text{ is regular (for suitable } m \text{ and } n)$
- ▶ $\varphi = \varphi_1 \vee \varphi_2$ with $FV(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$

$L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

applications of inverse homomorphism drop_i^{-1} to $L_a(\varphi_1)$ and $L_a(\varphi_2)$ yield regular sets $L_1, L_2 \subseteq (\{0, 1\}^{m+n})^*$ such that $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{m,n})$

Example

$\varphi = x < y \vee X(x)$ with $FV(\varphi) = (x, y, X)$

$$\triangleright L_a(x < y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^*$$

$$\triangleright L_a(X(x)) = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^*$$

$$\triangleright L_1 = \text{drop}_3^{-1} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \right) = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \\ * \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \\ * \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}^*$$

$$\triangleright L_2 = \text{drop}_2^{-1} \left(\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^* \right) = \left[\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ * \\ 1 \end{pmatrix} \right]^* \begin{pmatrix} 1 \\ * \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ * \\ 1 \end{pmatrix} \right]^*$$

$$\triangleright L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{2,1})$$

Definition

homomorphism

$$\text{drop}_i: (\{0,1\}^k)^* \rightarrow (\{0,1\}^{k-1})^*$$

is defined for $1 \leq i \leq k$ by dropping i -th component from vectors in $\{0,1\}^k$

$$\text{drop}_i \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_k \end{pmatrix}$$

Lemmata

- ▶ $A \subseteq (\{0,1\}^k)^*$ is regular $\implies \text{drop}_i(A) \subseteq (\{0,1\}^{k-1})^*$ is regular
- ▶ $B \subseteq (\{0,1\}^{k-1})^*$ is regular $\implies \text{drop}_i^{-1}(B) \subseteq (\{0,1\}^k)^*$ is regular

Proof (cont'd)

- ▶ $\varphi = \exists x. \psi \implies L_a(\psi)$ is regular
 - $\implies \text{drop}_i(L_a(\psi))$ is regular where i is position of x in $\text{FV}(\psi)$
 - $\implies L_a(\varphi) = \text{stz}(\text{drop}_i(L_a(\psi)))$ is regular
- ▶ $\varphi = \exists X. \psi \implies L_a(\psi)$ is regular
 - $\implies \text{drop}_i(L_a(\psi))$ is regular where i is position of X in $\text{FV}(\psi)$
 - $\implies L_a(\varphi) = \text{stz}(\text{drop}_i(L_a(\psi)))$ is regular

Example

$\varphi = \exists X. \psi$ with $\psi = X(x) \wedge \exists y. x < y \wedge X(y)$

- ▶ $L_a(\psi) = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^*$
- ▶ $\text{drop}_2(L_a(\psi)) = (0 + 0)^* 1(0 + 0)^* 0(0 + 0)^* = 0^* 100^* \neq 0^* 10^* = L_a(\varphi)$

Definition

$\text{stz}(A) = \{x \mid x\mathbf{0} \cdots \mathbf{0} \in A\}$ for $A \subseteq (\{0, 1\}^{m+n})^*$

"shorten trailing zeros"

Lemma

$A \subseteq (\{0, 1\}^k)^*$ is regular $\implies \text{stz}(A)$ is regular

Proof

- ▶ DFA $M = (Q, \Sigma, \delta, s, F)$ with $L(M) = A$
- ▶ construct DFA $M' = (Q, \Sigma, \delta, s, F')$ with $F' = \{q \in Q \mid \widehat{\delta}(q, x) \in F \text{ for some } x \in \mathbf{0}^*\}$
- ▶ $L(M') = \text{stz}(A)$

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$ using regularity preserving operations

Procedure

- ① eliminate assignments which do not correspond to string in Σ^*
- ② map strings in 0^*10^* to elements of Σ using homomorphism h
 $h: \{0^k10^l \mid k+1+l = |\Sigma|\} \rightarrow \Sigma$ which maps 0^k10^l to $k+1$ -th element of Σ

Lemma

$L(\varphi) = h(L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*)$ is regular

Corollary

WMSO definable sets are regular

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MONA

- ▶ MONA is state-of-the-art tool that implements decision procedures for **WS1S** and WS2S
- ▶ WS1S is weak monadic second-order theory of 1 successor = WMSO
- ▶ MONA translates WS1S formulas into minimum-state DFAs
- ▶ MONA confirms validity or produces counterexample

Demo

<https://www.brics.dk/mona/>

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Ebbinghaus, Flum and Thomas

- ▶ Section 10.9 of **Einführung in die mathematische Logik** (Springer Spektrum 2018)

Klarlund and Møller

- ▶ Section 3 of **MONA Version 1.4 User Manual** (2001)

Important Concepts

- | | | |
|-------------------|-------------------|--------|
| ▶ drop_i | ▶ m -admissible | ▶ stz |
| ▶ $L_a(\varphi)$ | ▶ MONA | ▶ WS1S |

homework for November 14