universität innsbruck

25W lecture 7

Automata and Logic

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Outline

1. **Summary of Previous Lecture**
2. **Presburger Arithmetic**
3. **Intermezzo**
4. **Presburger Arithmetic**
5. **WMSO**
6. **Further Reading**

Definitions

- $\mathrm{FV}(\varphi)$ denotes list of free variables in φ in fixed order with first–order variables preceding second–order ones
- assignment for φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is tuple $(i_1, \dots, i_m, I_1, \dots, I_n)$ such that $i_1, \dots, i_m$ are elements of $\mathbb{N}$ and $I_1, \dots, I_n$ are finite subsets of $\mathbb{N}$
- assignments are identified with strings over $\{0,1\}^{m+n}$
- string over $\{0,1\}^{m+n}$ is *m–admissible* if first m rows contain exactly one 1 each

Remarks

- every m–admissible string x induces assignment $\underline{x}$
- every assignment is induced by (not necessarily unique) m–admissible string: if x is m–admissible then $x\mathbf{0}$ is m–admissible and $\underline{x} = \underline{x\mathbf{0}}$

Lemma

set of m–admissible strings over $\{0,1\}^{m+n}$ is regular and accepted by DFA $\mathcal{A}_{m,n}$

Definition

$L_a(\varphi) = \{x \in (\{0,1\}^{m+n})^* \mid x \text{ is } m\text{–admissible and } \underline{x} \vDash \varphi\}$

Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

Definitions

- homomorphism $\text{drop}_i : (\{0,1\}^k)^* \to (\{0,1\}^{k-1})^*$ is defined for $1 \leqslant i \leqslant k$ by dropping i–th component from vectors in $\{0,1\}^k$
- $\text{stz}(A) = \{x \mid x\mathbf{0}\cdots\mathbf{0} \in A\} \supseteq A$ for $A \subseteq (\{0,1\}^{m+n})^*$ "shorten trailing zeros"

Lemma

$A \subseteq (\{0,1\}^k)^*$ is regular $\implies$ $\mathrm{stz}(A)$ is regular

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with $\mathrm{FV}(\varphi) = \{P_a \mid a \in \Sigma^*\}$ using regularity preserving operations

Procedure

① eliminate assignments which do not correspond to string in Σ^*

② map strings in 0^*10^* to elements of Σ using homorphism $h\colon \{0,1\}^{|\Sigma|} \to \Sigma$ which maps 0^k10^l to $k{+}1$-th element of Σ

Lemma

$L(\varphi) = h(L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*)$ is regular

Corollary

WMSO definable sets are regular

MONA

- MONA is state–of–the–art tool that implements decision procedures for WS1S and WS2S
- WS1S is weak monadic second–order theory of 1 successor = WMSO

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

Outline

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formulas of Presburger arithmetic

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$$t ::= 0 \mid 1 \mid t_1 + t_2 \mid x$$

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Examples

1. $\exists\, y.\, x = y + y + y + 1$

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$$\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi) \qquad \varphi \to \psi := \neg\varphi \vee \psi \qquad \top := \neg\bot$$

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$$\mathrm{n} \ := \ \underbrace{1 + \cdots + 1}_{n} \qquad \mathrm{n}\, x \ := \ \underbrace{x + \cdots + x}_{n} \quad \text{for } n > 1$$

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Remark

$t_1 < t_2$ can be modeled as $\exists x.\, x \neq 0 \wedge t_1 + x = t_2$

Remark

every $t_1 = t_2$ can be written as $a_1x_1 + \cdots + a_nx_n = b$ with $a_1, \ldots, a_n, b \in \mathbb{Z}$

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$$\psi(0) \wedge \forall x.\, (\psi(x) \rightarrow \psi(x+1)) \rightarrow \forall x.\, \psi(x)$$

  for every formula $\psi(x)$ with single free variable x

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  for every formula $\psi(x)$ with single free variable x
- $\forall x.\, x + 0 = x$
- $\forall x.\, \forall y.\, x + (y + 1) = (x + y) + 1$

Example

given natural numbers $a_1, \ldots, a_n > 0$

$$(\forall y. x < y \rightarrow \exists x_1. \ldots \exists x_n. a_1 x_1 + \cdots + a_n x_n = y) \wedge \neg(\exists x_1. \ldots \exists x_n. a_1 x_1 + \cdots + a_n x_n = x)$$

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expresses largest number x that does not satisfy $a_1x_1 + \cdots + a_nx_n = x$ for some $x_1, \ldots, x_n \in \mathbb{N}$

Example (Frobenius Coin Problem)

given natural numbers $a_1, \ldots, a_n > 0$

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Theorem (Presburger 1929)

Presburger arithmetic is decidable

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Decision Procedures

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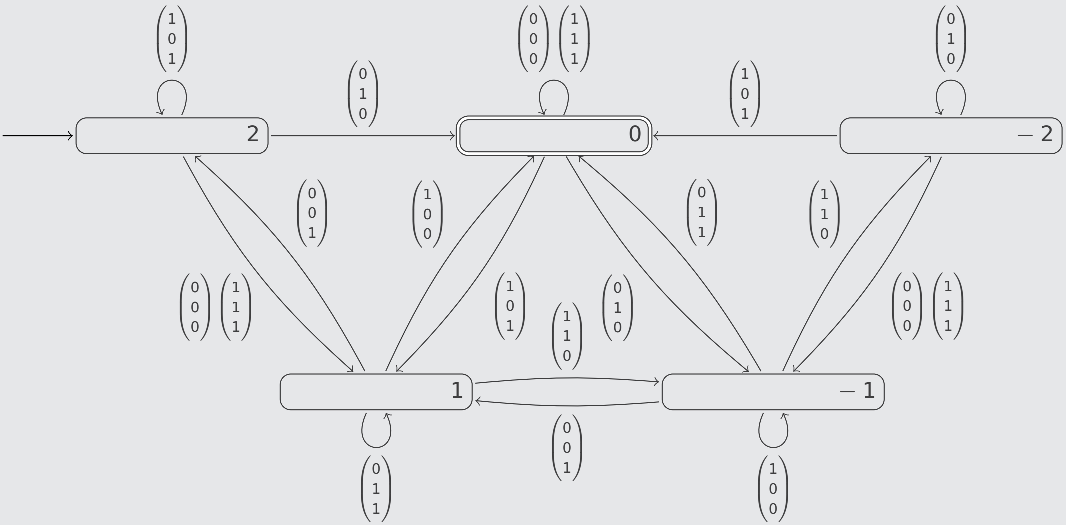
Example

Presburger arithmetic formula $\varphi\colon\ x + 2y - 3z = 2$

Example (cont'd)



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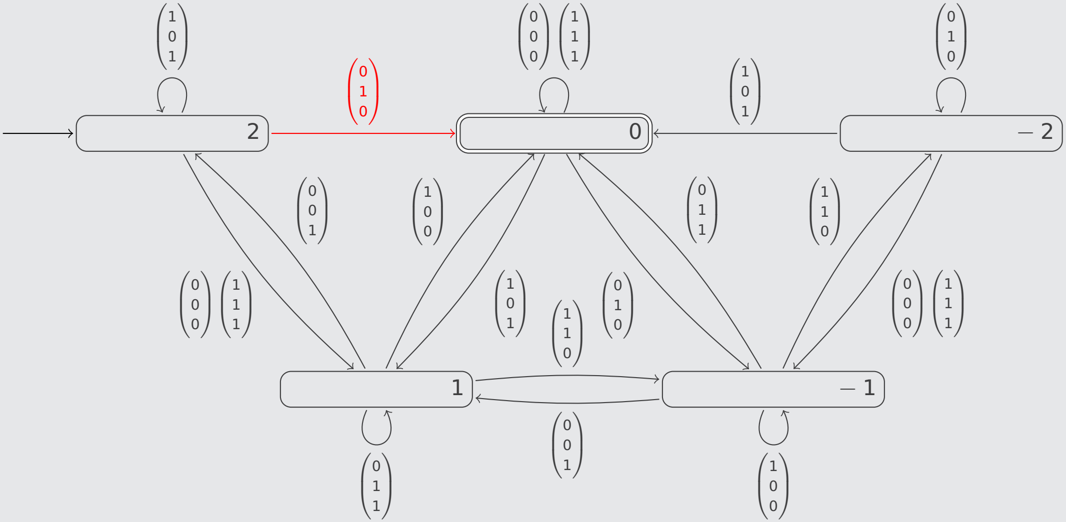

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Presburger arithmetic formula φ: $x + 2y - 3z = 2$

some accepted strings:

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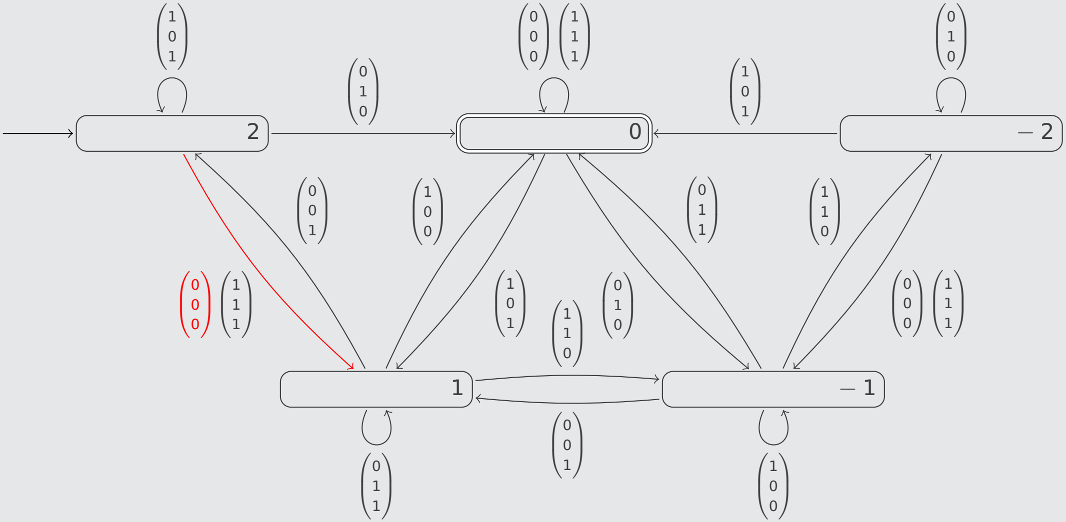

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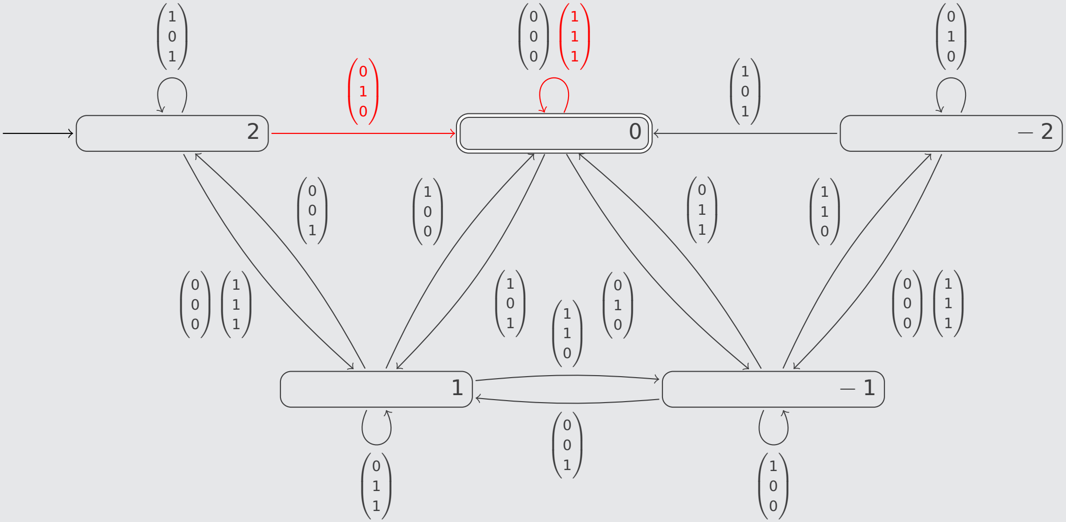

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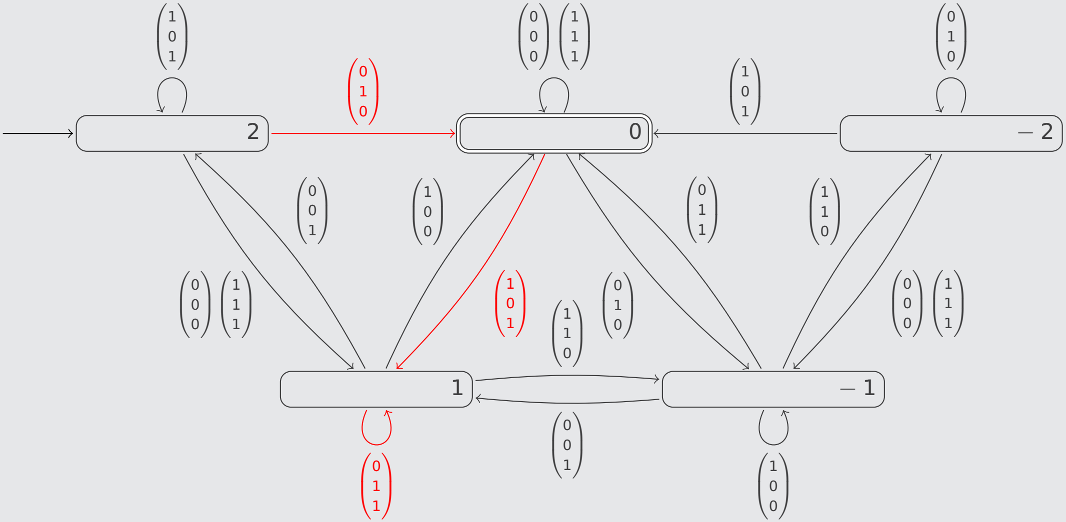

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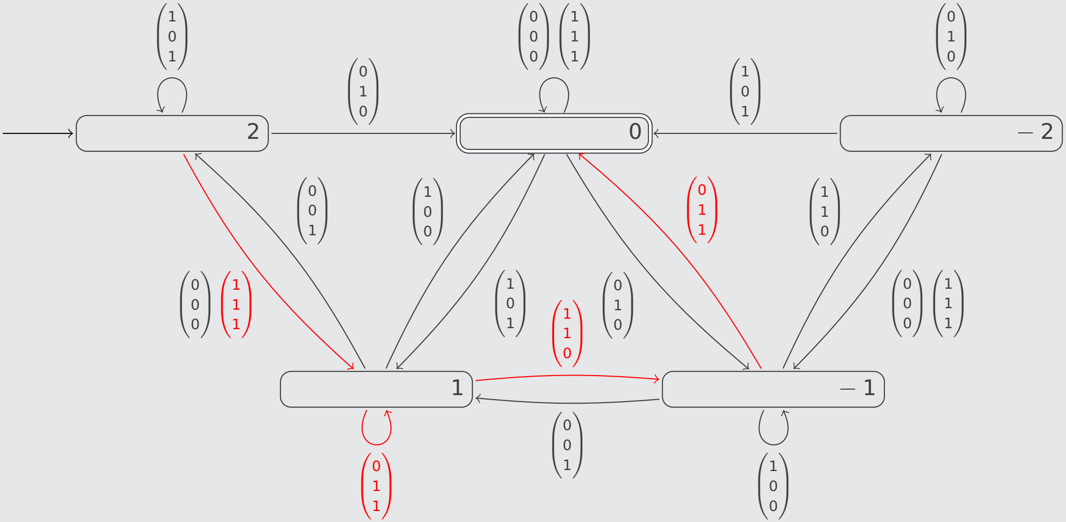

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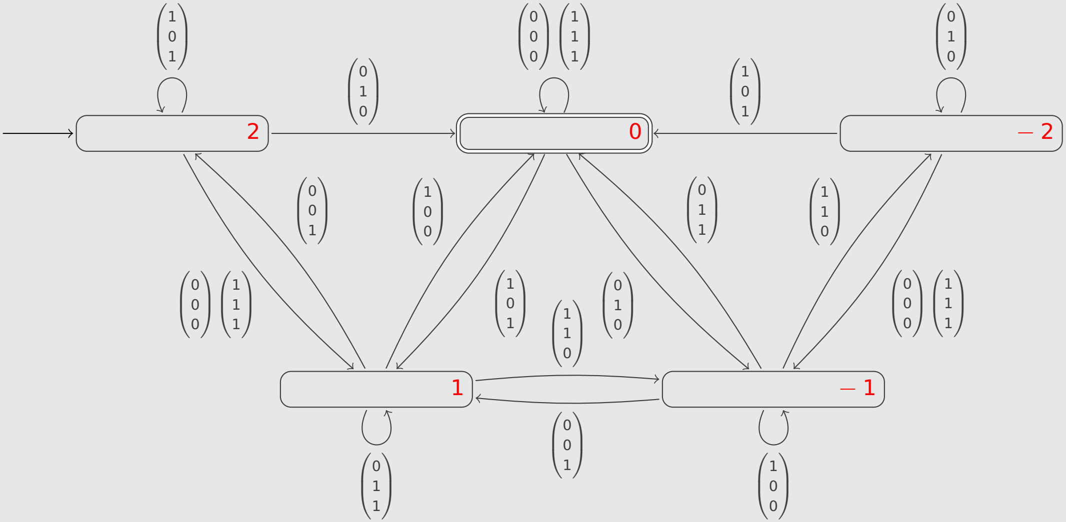

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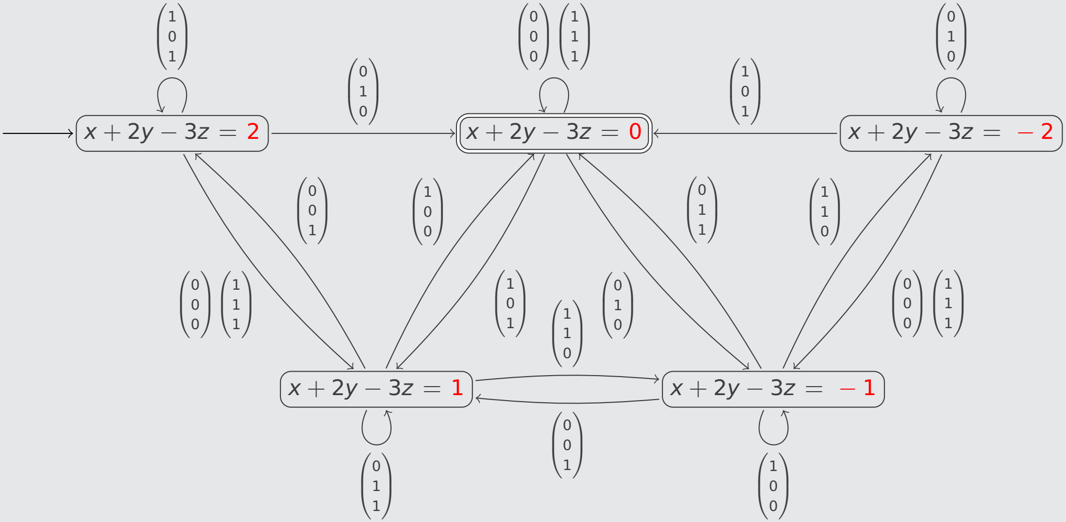

Definition (Representation)

- sequence of n natural numbers is represented as string over

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- $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ is represented by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Definition (Representation)

- sequence of n natural numbers is represented as string over

$$\Sigma_n = \{(b_1 \cdots b_n)^{\mathsf{T}} \mid b_1, \ldots, b_n \in \{0,1\}\}$$

- $x = \begin{pmatrix} b_1^1 \\ \vdots \\ b_n^1 \end{pmatrix} \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix} \cdots \begin{pmatrix} b_1^m \\ \vdots \\ b_n^m \end{pmatrix} \in \Sigma_n^*$ represents $x_1 = (b_1^m \cdots b_1^2 b_1^1)_2, \ \ldots, \ x_n = (b_n^m \cdots b_n^2 b_n^1)_2$
- $\underline{x} = (x_1, \ldots, x_n)$

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Definition

for Presburger arithmetic formula φ with $\mathrm{FV}(\varphi) = (x_1, \ldots, x_n)$

$$L(\varphi) = \{x \in \Sigma_n^* \mid \underline{x} \vDash \varphi\}$$

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Theorem (Presburger 1929)

Presburger arithmetic is decidable

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Proof Sketch

- construct finite automaton A_φ for every Presburger arithmetic formula φ

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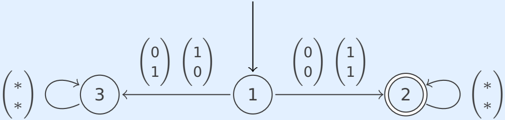
- construct finite automaton A_φ for every Presburger arithmetic formula φ
- induction on φ
- $L(A_\varphi) = L(\varphi)$

Outline

Question

Consider the following automaton A:



For which of the following formulas φ does $L(A) = L(\varphi)$ hold ?

A $\neg(x = y)$

B $x + y > 0$

C $\exists z. x + y = 2z$

D $(\exists z. x = 2z \wedge \forall z. \neg(y = 2z)) \vee (\exists z. y = 2z \wedge \forall z. \neg(x = 2z))$

Outline

Definition (Automaton for Atomic Formula)

DFA $A_\varphi = (Q, \Sigma_n, \delta, s, F)$ for $\varphi(x_1, \ldots, x_n)$: $a_1x_1 + \cdots + a_nx_n = b$

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if $\delta(i, (b_1 \cdots b_n)^\mathsf{T}) = j$ then $a_1 x_1 + \cdots + a_n x_n = j \iff a_1(2x_1 + b_1) + \cdots + a_n(2x_n + b_n) = i$

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$\varphi(x, y)$: $x + 2y = 3$

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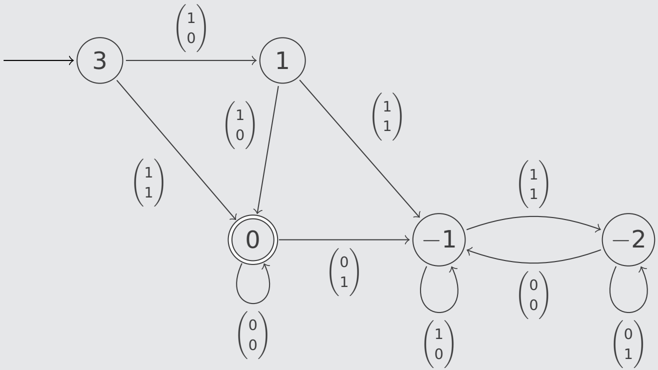

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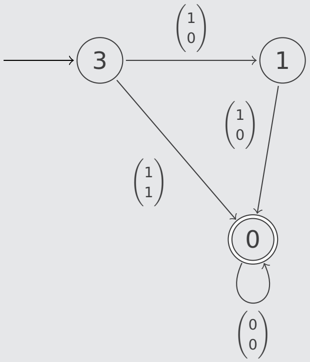

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$$|i - (a_1 b_1 + \cdots + a_n b_n)| \leqslant |i| + |a_1 b_1| + \cdots + |a_n b_n|$$

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$$\delta(i, b') = \frac{i - (a_1 b_1 + \cdots + a_n b_n)}{2} \in Q$$

$$\begin{aligned}
|i - (a_1 b_1 + \cdots + a_n b_n)| &\leqslant |i| + |a_1 b_1| + \cdots + |a_n b_n| \\
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Outline

Boolean Operations

boolean operation	automata construction
$\neg$	complement

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Example

Presburger arithmetic formula

$$\neg((x + 2y - 3z \neq 2 \ \wedge \ 2x - y + z = 3) \ \vee \ x - 3y - z \neq 1)$$

Boolean Operations

boolean operation	automata construction
$\neg$	complement $\quad C$
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$\vee$	union

Example

Presburger arithmetic formula

$$\neg((x + 2y - 3z \neq 2 \ \wedge \ 2x - y + z = 3) \ \vee \ x - 3y - z \neq 1))$$

is implemented as

$$C(A_{x+2y-3z=2})$$

Boolean Operations

boolean operation	automata construction	
$\neg$	complement	C
$\wedge$	intersection	I
$\vee$	union	

Example

Presburger arithmetic formula

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Presburger arithmetic formula

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Presburger arithmetic formula

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Example

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- before intersection can be computed $A_{x+2y=3}$ needs to operate on Σ_3

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- before intersection can be computed $A_{x+2y=3}$ needs to operate on Σ_3

Definition (Cylindrification)

$\mathrm{C}_i(R) \subseteq \Sigma_{n+1}^*$ is defined for $R \subseteq \Sigma_n^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\mathrm{C}_i(R) = \left\{ x_1 \cdots x_m \in \Sigma_{n+1}^* \;\middle|\; \mathrm{drop}_i(x_1) \cdots \mathrm{drop}_i(x_m) \in R \right\}$$

with $\mathrm{drop}_i\big((b_1 \cdots b_{n+1})^\mathsf{T}\big) = (b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1})^\mathsf{T}$

Example

- $L(A_{x+2y=3}) = \left\{ \binom{1}{1}, \binom{1}{0}\binom{1}{0} \right\} \left\{ \binom{0}{0} \right\}^*$

Example

- $L(A_{x+2y=3}) = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}^*$
- $L(C_3(A_{x+2y=3})) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

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Lemma

if $R \subseteq \Sigma_n^*$ is regular then $C_i(R) \subseteq \Sigma_{n+1}^*$ is regular for every $1 \leqslant i \leqslant n+1$

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Remark

- drop_i is homomorphism from Σ_{n+1}^* to Σ_n^*

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Remark

- drop_i is homomorphism from Σ_{n+1}^* to Σ_n^*
- $\mathrm{C}_i(R) = \mathrm{drop}_i^{-1}(R)$

Outline

Definition (Projection)

$\Pi_i(R) \subseteq \Sigma_n^*$ is defined for $R \subseteq \Sigma_{n+1}^*$ and index $1 \leqslant i \leqslant n+1$ as

$$\Pi_i(R) = \{\mathrm{drop}_i(x_1) \cdots \mathrm{drop}_i(x_m) \in \Sigma_n^* \mid x_1 \cdots x_m \in R\}$$

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Example

1. solutions of $\exists\, y.\, x + 2y - 3z = 2$ correspond to $\mathrm{stz}(\Pi_2(A_{x+2y-3z=2}))$

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2. solutions of $\forall\, y.\, x + 2y - 3z = 2$ correspond to $\mathrm{C}(\mathrm{stz}(\Pi_2(\mathrm{C}(A_{x+2y-3z=2}))))$

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$\Pi_i(R) \subseteq \Sigma_n^*$ is defined for $R \subseteq \Sigma_{n+1}^*$ and index $1 \leqslant i \leqslant n+1$ as

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1. solutions of $\exists\, y.\, x + 2y - 3z = 2$ correspond to $\mathrm{stz}(\Pi_2(A_{x+2y-3z=2}))$
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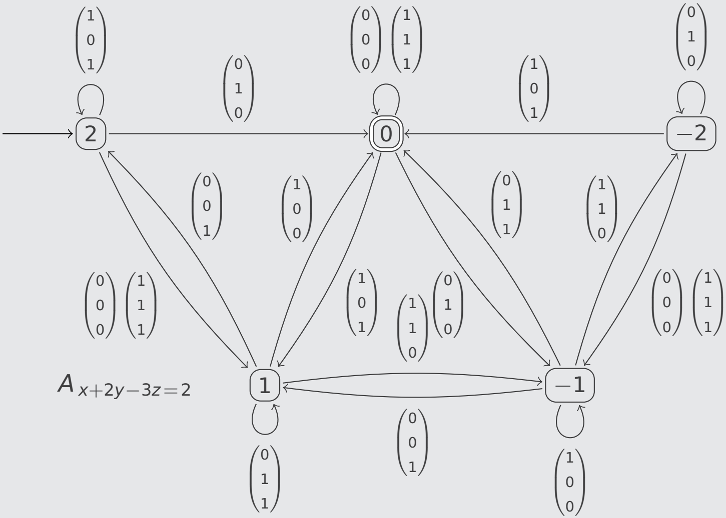
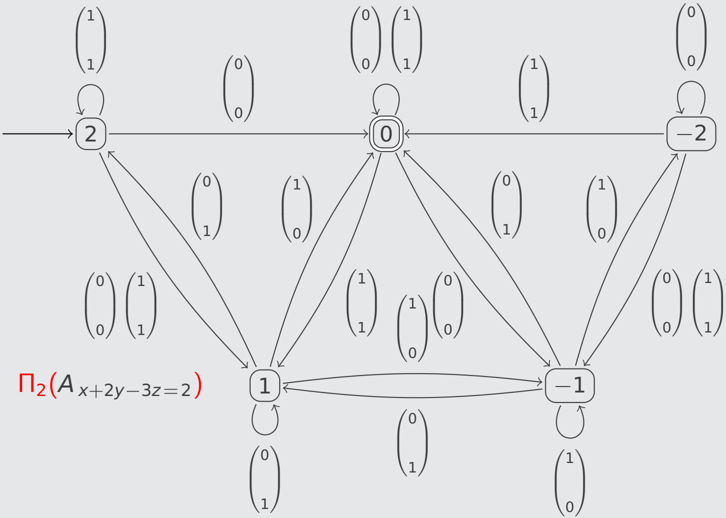
Example



Example



$\Pi_2(A_{x+2y-3z=2})$

Outline

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Decision Procedures

- quantifier elimination
- automata techniques
- translation to WMSO

Procedure

- map variables in Presburger arithmetic formula to second−order variables in WMSO

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Example

26 is represented by $\{1,3,4\}$ since $(26)_2 = 11010$

Procedure

- map variables in Presburger arithmetic formula to second−order variables in WMSO
- n is represented as set of "1" positions in reverse binary notation of n
- 0 and 1 in Presburger arithmetic formulas are translated into ZERO and ONE with

$$\forall\, x.\, \neg \mathrm{ZERO}(x) \qquad\qquad \forall\, x.\, \mathrm{ONE}(x) \leftrightarrow x = 0$$

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Procedure

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- n is represented as set of "1" positions in reverse binary notation of n
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$$\forall x.\, \neg \mathrm{ZERO}(x) \qquad\qquad \forall x.\, \mathrm{ONE}(x) \leftrightarrow x = 0$$

- $+$ in Presburger arithmetic formula is translated into ternary predicate P_+ with

$$\begin{aligned} P_+(X,Y,Z) \;:=\; \exists\, C.\, \neg C(0) \wedge \big(\forall x.\, C(x+1) \leftrightarrow X(x) \wedge Y(x) \vee X(x) \wedge C(x) \vee Y(x) \wedge C(x)\big) \wedge \\ \big(\forall x.\, Z(x) \leftrightarrow X(x) \wedge Y(x) \wedge C(x) \vee X(x) \wedge \neg Y(x) \wedge \neg C(x) \vee \\ \neg X(x) \wedge Y(x) \wedge \neg C(x) \vee \neg X(x) \wedge \neg Y(x) \wedge C(x)\big) \end{aligned}$$

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Example

Presburger arithmetic formula $\exists\, y.\, x = y + y + 1$ is transformed into WMSO formula

$$\big(\forall\, x.\, \mathrm{ONE}(x) \leftrightarrow x = 0\big) \;\wedge\; \exists\, Y.\, \exists\, Z.\, P_+(Y,Y,Z) \;\wedge\; P_+(Z,\mathrm{ONE},X)$$

Example

Presburger arithmetic formula $\exists\, y.\, x = y + y + 1$ is transformed into WMSO formula

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Corollary

Presburger arithmetic is decidable

Outline

1. Summary of Previous Lecture
2. Presburger Arithmetic
3. Intermezzo
4. Presburger Arithmetic
5. WMSO
6. **Further Reading**

Boudet and Comon

- Diophantine Equations, Presburger Arithmetic and Finite Automata, Proc. 21st International Colloquium on Trees in Algebra and Programming, LNCS 1059, pp. 30 – 43, 1996

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Esparza and Blondin

- Chapter 9 of Automata Theory: An Algorithmic Approach (MIT Press 2023)

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Important Concepts

- A_φ
- $L(\varphi)$
- cylindrification
- projection
- Presburger arithmetic

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homework for November 21