



Automata and Logic

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Outline

- 1. Summary of Previous Lecture**
- 2. Infinite Strings**
- 3. Büchi Automata**
- 4. Intermezzo**
- 5. Closure Properties**
- 6. Further Reading**

Definition

formulas of **Presburger arithmetic**

$$\varphi ::= \perp \mid \neg \varphi \mid \varphi_1 \vee \varphi_2 \mid \exists x. \varphi \mid t_1 = t_2 \mid t_1 < t_2$$

$$t ::= 0 \mid 1 \mid t_1 + t_2 \mid x$$

Abbreviations

$$\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$$

$$\varphi \rightarrow \psi := \neg\varphi \vee \psi$$

$$\top := \neg\perp$$

$$\forall x. \varphi := \neg\exists x. \neg\varphi$$

$$t_1 \leq t_2 := t_1 < t_2 \vee t_1 = t_2$$

$$n := \underbrace{1 + \dots + 1}_n$$

$$nx := \underbrace{x + \dots + x}_n \quad \text{for } n > 1$$

Definitions

- ▶ assignment α is mapping from first-order variables to $\mathbb{N}$
- ▶ extension to terms: $\alpha(0) = 0 \quad \alpha(1) = 1 \quad \alpha(t_1 + t_2) = \alpha(t_1) + \alpha(t_2)$

Definition

assignment α satisfies formula φ ($\alpha \models \varphi$):

$$\alpha \not\models \perp$$

$$\alpha \models \neg \varphi \iff \alpha \not\models \varphi$$

$$\alpha \models \varphi_1 \vee \varphi_2 \iff \alpha \models \varphi_1 \text{ or } \alpha \models \varphi_2$$

$$\alpha \models \exists x. \varphi \iff \alpha[x \mapsto n] \models \varphi \text{ for some } n \in \mathbb{N}$$

$$\alpha \models t_1 = t_2 \iff \alpha(t_1) = \alpha(t_2)$$

$$\alpha \models t_1 < t_2 \iff \alpha(t_1) < \alpha(t_2)$$

Remark

every $t_1 = t_2$ can be written as $a_1x_1 + \dots + a_nx_n = b$ with $a_1, \dots, a_n, b \in \mathbb{Z}$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Decision Procedures

- ▶ quantifier elimination
- ▶ automata techniques
- ▶ translation to WMSO

Definition (Representation)

- ▶ sequence of n natural numbers is represented as string over

$$\Sigma_n = \{(b_1 \cdots b_n)^T \mid b_1, \dots, b_n \in \{0, 1\}\}$$

- ▶ $x = \begin{pmatrix} b_1^1 \\ \vdots \\ b_n^1 \end{pmatrix} \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix} \cdots \begin{pmatrix} b_1^m \\ \vdots \\ b_n^m \end{pmatrix} \in \Sigma_n^*$ represents $x_1 = (b_1^m \cdots b_1^2 b_1^1)_2, \dots, x_n = (b_n^m \cdots b_n^2 b_n^1)_2$
- ▶ $\underline{x} = (x_1, \dots, x_n)$

Definition

for Presburger arithmetic formula φ with $FV(\varphi) = (x_1, \dots, x_n)$

$$L(\varphi) = \{x \in \Sigma_n^* \mid \underline{x} \models \varphi\}$$

Theorem (Presburger 1929)

Presburger arithmetic is decidable

Proof Sketch

- ▶ construct finite automaton A_φ for every Presburger arithmetic formula φ
- ▶ induction on φ
- ▶ $L(A_\varphi) = L(\varphi)$

Definition (Automaton for Atomic Formula)

finite automaton $A_\varphi = (Q, \Sigma_n, \delta, s, F)$ for $\varphi(x_1, \dots, x_n): a_1x_1 + \dots + a_nx_n = b$

► $Q \subseteq \{i \mid |i| \leq |b| + |a_1| + \dots + |a_n|\} \cup \{\perp\}$

► $\delta(i, (b_1 \dots b_n)^T) = \begin{cases} \frac{i - (a_1b_1 + \dots + a_nb_n)}{2} & \text{if } i - (a_1b_1 + \dots + a_nb_n) \text{ is even} \\ \perp & \text{if } i - (a_1b_1 + \dots + a_nb_n) \text{ is odd or } i = \perp \end{cases}$

► $s = b$

► $F = \{0\}$

Lemma

if $\delta(i, (b_1 \dots b_n)^T) = j$ then $a_1x_1 + \dots + a_nx_n = j \iff a_1(2x_1 + b_1) + \dots + a_n(2x_n + b_n) = i$

Theorem

► A_φ is well-defined

► $L(A_\varphi) = L(\varphi)$

Boolean Operations

boolean operation	automata construction	
$\neg$	complement	C
$\wedge$	intersection	I
$\vee$	union	U

Definition (Cylindrification)

$C_i(R) \subseteq \Sigma_{n+1}^*$ is defined for $R \subseteq \Sigma_n^*$ and index $1 \leq i \leq n+1$ as

$$C_i(R) = \{x_1 \cdots x_m \in \Sigma_{n+1}^* \mid \text{drop}_i(x_1) \cdots \text{drop}_i(x_m) \in R\}$$

with $\text{drop}_i((b_1 \cdots b_{n+1})^T) = (b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1})^T$

Lemma

if $R \subseteq \Sigma_n^*$ is regular then $C_i(R) \subseteq \Sigma_{n+1}^*$ is regular for every $1 \leq i \leq n+1$

Definition (Projection)

$\Pi_i(R) \subseteq \Sigma_n^*$ is defined for $R \subseteq \Sigma_{n+1}^*$ and index $1 \leq i \leq n+1$ as

$$\Pi_i(R) = \{\text{drop}_i(x_1) \cdots \text{drop}_i(x_m) \in \Sigma_n^* \mid x_1 \cdots x_m \in R\}$$

Lemma

if $R \subseteq \Sigma_{n+1}^*$ is regular then $\Pi_i(R) \subseteq \Sigma_n^*$ is regular for every $1 \leq i \leq n+1$

Translation from Presburger Arithmetic to WMSO

- ▶ map variables in Presburger arithmetic formula to second-order variables in WMSO
- ▶ n is represented as set of "1" positions in reverse binary notation of n
- ▶ 0 and 1 in Presburger arithmetic formulas are translated into ZERO and ONE with

$$\forall x. \neg \text{ZERO}(x)$$

$$\forall x. \text{ONE}(x) \leftrightarrow x = 0$$

- ▶ $+$ in Presburger arithmetic formula is translated into ternary predicate P_+ with

$$\begin{aligned} P_+(X, Y, Z) := & \exists C. \neg C(0) \wedge (\forall x. C(x+1) \leftrightarrow X(x) \wedge Y(x) \vee X(x) \wedge C(x) \vee Y(x) \wedge C(x)) \wedge \\ & (\forall x. Z(x) \leftrightarrow X(x) \wedge Y(x) \wedge C(x) \vee X(x) \wedge \neg Y(x) \wedge \neg C(x) \vee \\ & \neg X(x) \wedge Y(x) \wedge \neg C(x) \vee \neg X(x) \wedge \neg Y(x) \wedge C(x)) \end{aligned}$$

Automata

- ▶ (deterministic, nondeterministic, alternating) finite automata
- ▶ regular expressions
- ▶ (alternating) **Büchi automata**

Logic

- ▶ (weak) monadic second-order logic
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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Example

$$x(i) = \begin{cases} a & \text{if } i \text{ is even} \\ b & \text{if } i \text{ is odd} \end{cases}$$

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- ▶ $|x|_a$ for $x \in \Sigma^\omega$ and $a \in \Sigma$ denotes number of occurrences of a in x

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Remarks

- ▶ infinite string x is identified with infinite sequence $x(0)x(1)x(2) \cdots$
- ▶ $|x|_a = \infty$ for at least one $a \in \Sigma$

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- ▶ **left-concatenation** of $u \in \Sigma^*$ and $v \in \Sigma^\omega$ is denoted by $u \cdot v \in \Sigma^\omega$

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- ▶ **left-concatenation** of $U \subseteq \Sigma^*$ and $V \subseteq \Sigma^\omega$

$$U \cdot V = \{u \cdot v \mid u \in U \text{ and } v \in V\}$$

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- ▶ $\sim V = \Sigma^\omega - V$ is complement of $V \subseteq \Sigma^\omega$
- ▶ $U^\omega = \{u_0 \cdot u_1 \cdot \dots \mid u_i \in U - \{\epsilon\} \text{ for all } i \in \mathbb{N}\}$ is ω -iteration of $U \subseteq \Sigma^*$

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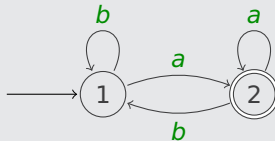
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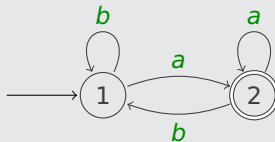
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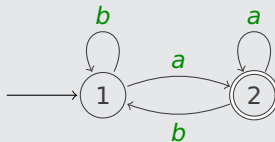
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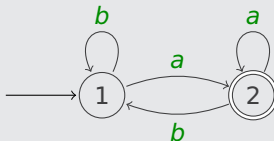
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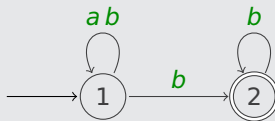
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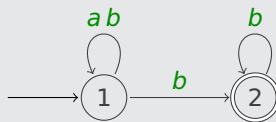
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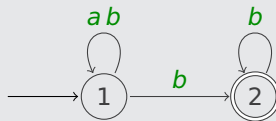
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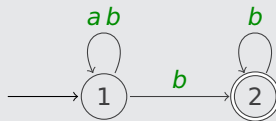
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► $L(M) = \{x \in \{a, b\}^\omega \mid |x|_a \neq \infty\} = (a + b)^* b^\omega$

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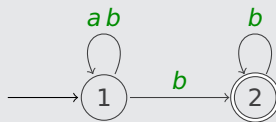
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Definitions

- ▶ set $A \subseteq \Sigma^\omega$ is **ω -regular** if $A = L(M)$ for some NBA M

Example

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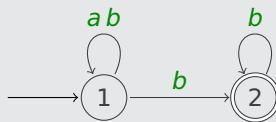
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- ▶ **deterministic Büchi automaton (DBA)** is NBA $(Q, \Sigma, \Delta, S, F)$ with
 - ① $|S| = 1$
 - ② $|\Delta(q, a)| = 1$ for all $q \in Q$ and $a \in \Sigma$

Example

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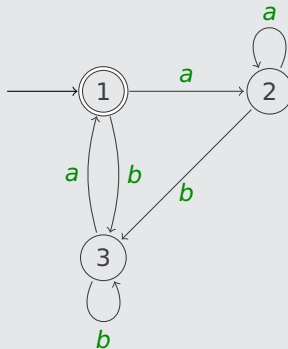
- ▶ $L(M) = \{x \in \{a, b\}^\omega \mid |x|_a \neq \infty\} = (a + b)^* b^\omega$
- ▶ M is not deterministic

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Example

$\{x \in \{a, b\}^\omega \mid |x|_a = |x|_b = \infty\}$ is accepted by DBA



Theorem

not every ω -regular set is accepted by DBA

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$$\text{let } l_1 = i_1 - i_0 - 1$$

$$x_2 = b^{i_0} a b^{l_1} a b^\omega$$

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► suppose $L = L(M)$ for DBA $M = (Q, \Sigma, \Delta, S, F)$

$$x_0 = b^\omega \in L \quad \implies \quad \exists \text{ accepting run } q_0, q_1, \dots \quad \implies \quad \exists i_0 \geq 0 \quad \text{with } q_{i_0} \in F$$

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► $x = b^{i_0} ab^{l_1} \dots ab^{l_j} (ab^{l_j+1} \dots ab^{l_k})^\omega$

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Lemma

every ω -regular set is accepted by NBA with one start state

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- ▶ $A = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$
- ▶ define NBA $N = (Q', \Sigma, \Delta', \{s\}, F)$ with $Q' = Q \uplus \{s\}$ and

$$\Delta'(p, a) = \begin{cases} \Delta(p, a) & \text{if } p \neq s \\ \{q \in Q \mid q \in \Delta(p', a) \text{ for some } p' \in S\} & \text{if } p = s \end{cases}$$

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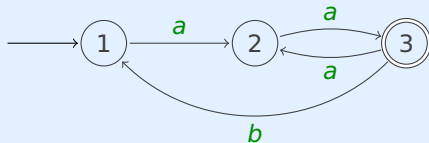
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Outline

1. Summary of Previous Lecture
2. Infinite Strings
3. Büchi Automata
- 4. Intermezzo**
5. Closure Properties
6. Further Reading

Question

Which statement about the following NBA M is true ?



- A** $L(M) = \emptyset$
- B** $(b^*aa)^\omega \in L(M)$
- C** $L(M) = \{x \mid |x|_a = \infty \text{ and } |x|_b \neq \infty\}$
- D** $L(M) = \{x \in \{a,b\}^\omega \mid \text{every } b \text{ follows an even number of } a\text{'s}\}$



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Theorem

ω -regular sets are effectively closed under union

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Proof (construction)

- ▶ $A = L(M_1)$ for NBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
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Theorem

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product construction needs to be modified

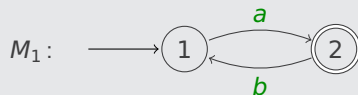
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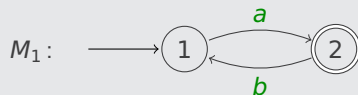
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$$L(M_1) = a(ba)^\omega = (ab)^\omega$$

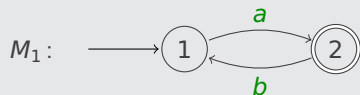
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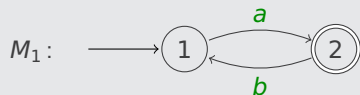
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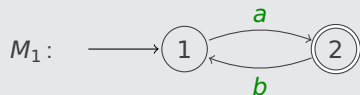
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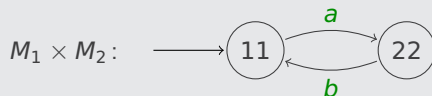


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$$j = \begin{cases} 1 & \text{if } i = 0 \text{ and } p' \in F_1 \text{ or } i = 1 \text{ and } q' \notin F_2 \\ 2 & \text{otherwise} \end{cases}$$

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② $S = S_1 \times S_2 \times \{0\}$

③ $F = Q_1 \times Q_2 \times \{2\}$

④ $\Delta((p, q, i), a) = \{(p', q', j) \mid p' \in \Delta_1(p, a) \text{ and } q' \in \Delta_2(q, a)\}$ with

$$j = \begin{cases} 1 & \text{if } i = 0 \text{ and } p' \in F_1 \text{ or } i = 1 \text{ and } q' \notin F_2 \\ 2 & \text{if } i = 1 \text{ and } q' \in F_2 \end{cases}$$

Theorem

ω -regular sets are effectively closed under intersection

Proof (modified product construction)

► $A = L(M_1)$ for NBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$

► $A \cap B = L(M)$ for NBA $M = (Q, \Sigma, \Delta, S, F)$ with

① $Q = Q_1 \times Q_2 \times \{0, 1, 2\}$

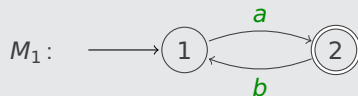
② $S = S_1 \times S_2 \times \{0\}$

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Example



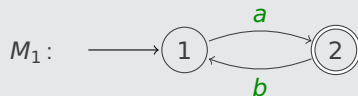
$$L(M_1) = a(ba)^\omega = (ab)^\omega$$



$$L(M_2) = (aa^*b)^\omega$$

$$L(M_1) \cap L(M_2) = (ab)^\omega$$

Example

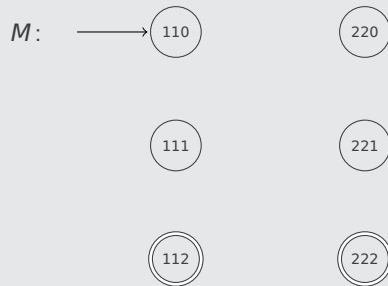


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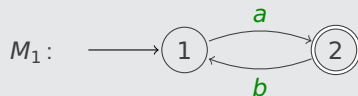


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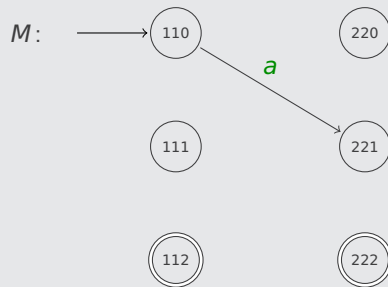


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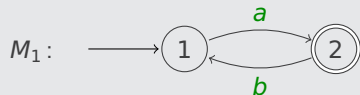


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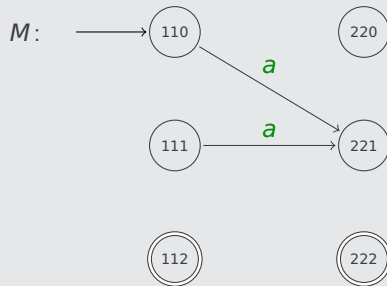


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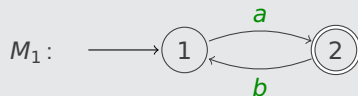


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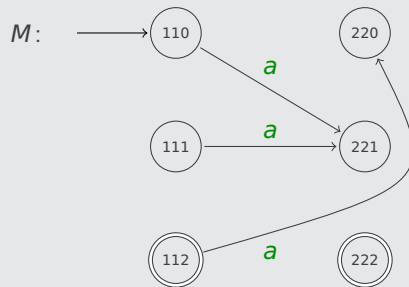


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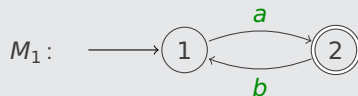


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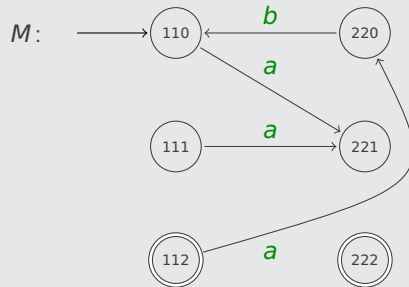


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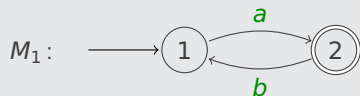


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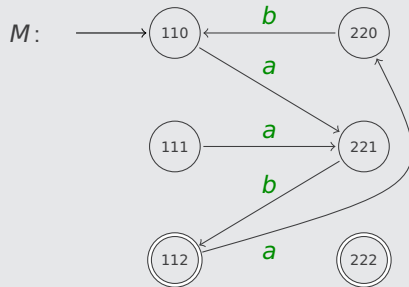


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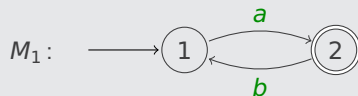


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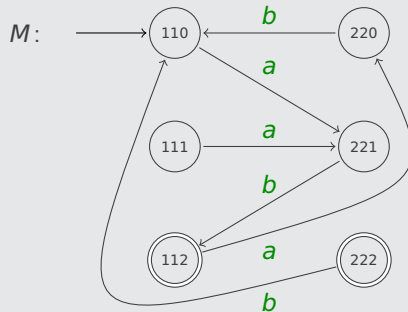


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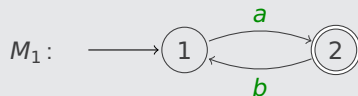


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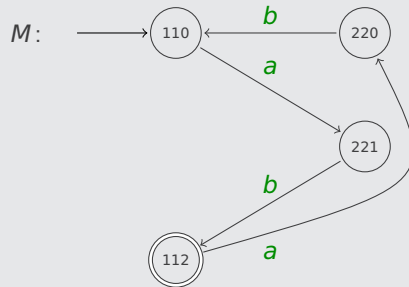


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Theorem

left-concatenation of regular set and ω -regular set is ω -regular

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Proof (construction)

► $A = L(M_1)$ for NFA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for NBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$

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 - ① $Q = Q_1 \cup Q_2$

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 - ③ $F = F_2$
 - ④ $\Delta = \Delta_1 \cup \Delta_2 \cup \{(p, a, q) \mid (p, a, f) \in \Delta_1 \text{ for some } f \in F_1 \text{ and } q \in S_2\}$

Theorem

ω -iteration of regular set is ω -regular

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Proof (construction)

► $A = L(M)$ for NFA $M = (Q, \Sigma, \Delta, S, F)$

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Proof (construction)

- ▶ $A = L(M)$ for NFA $M = (Q, \Sigma, \Delta, S, F)$
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Theorem

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Proof (construction)

- ▶ $A = L(M)$ for NFA $M = (Q, \Sigma, \Delta, S, F)$
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$$\Delta' = \Delta \cup \{(s, a, q) \mid (p, a, q) \in \Delta \text{ for some } p \in S\}$$

Theorem

ω -iteration of regular set is ω -regular

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- ▶ $L(M'') = L(M')^\omega$

Theorem

set $A \subseteq \Sigma^\omega$ is ω -regular $\iff$ $A = U_1 \cdot V_1^\omega \cup \dots \cup U_n \cdot V_n^\omega$
for some $n \geq 0$ and regular $U_1, \dots, U_n, V_1, \dots, V_n \subseteq \Sigma^*$

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Proof ($\Leftarrow$)

A is ω -regular using closure properties

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Proof ($\Leftarrow$)

A is ω -regular using closure properties: ω -iteration

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Proof ($\Leftarrow$)

A is ω -regular using closure properties: ω -iteration, left-concatenation

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Proof ($\Leftarrow$)

A is ω -regular using closure properties: ω -iteration, left-concatenation, union

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Proof ($\Leftarrow$)

A is ω -regular using closure properties: ω -iteration, left-concatenation, union

Proof ($\Rightarrow$)

► $A = L(M)$ for some NBA $M = (Q, \Sigma, \Delta, S, F)$

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Proof ($\Leftarrow$)

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Proof ($\Rightarrow$)

- ▶ $A = L(M)$ for some NBA $M = (Q, \Sigma, \Delta, S, F)$
- ▶ L_{pq} for $p, q \in Q$ is set of strings $x \in \Sigma^*$ such that $q \in \hat{\Delta}(\{p\}, x)$

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set $A \subseteq \Sigma^\omega$ is ω -regular $\iff$ $A = U_1 \cdot V_1^\omega \cup \dots \cup U_n \cdot V_n^\omega$
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- ▶ L_{pq} is regular for all $p, q \in Q$
- ▶ $A = \bigcup_{p \in S, q \in F} L_{pq} \cdot L_{qq}^\omega$

Outline

1. Summary of Previous Lecture

2. Infinite Strings

3. Büchi Automata

4. Intermezzo

5. Closure Properties

6. Further Reading

- ▶ Chapter 5 of **Automatentheorie und Logik** (Springer 2011)

Hofmann and Lange

- ▶ Chapter 5 of Automaten-theorie und Logik (Springer 2011)

Esparza and Blondin

- ▶ Chapter 10 of [Automata Theory: An Algorithmic Approach](#) (MIT Press 2023)

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Important Concepts

- | | | | |
|-------------------|----------------------|-------------------|-----------------------|
| ▶ Büchi automaton | ▶ left-concatenation | ▶ NBA | ▶ ω -iteration |
| ▶ DBA | | ▶ Σ^ω | ▶ ω -regular |

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homework for November 28