Automata and Logic

Aart Middeldorp and Samuel Frontull

Outline

1. **Summary of Previous Lecture**
2. **Complementation**
3. **Intermezzo**
4. **Monadic Second–Order Logic**
5. **Further Reading**

Definitions

- infinite string over alphabet Σ is function $x\colon \mathbb{N} \to \Sigma$
- Σ^ω denotes set of all infinite strings over Σ
- $|x|_a$ for $x \in \Sigma^\omega$ and $a \in \Sigma$ denotes number of occurrences of a in x
- left–concatenation of $u \in \Sigma^*$ and $v \in \Sigma^\omega$ is denoted by $u \cdot v \in \Sigma^\omega$
- left–concatenation of $U \subseteq \Sigma^*$ and $V \subseteq \Sigma^\omega$

$$U \cdot V = \{u \cdot v \mid u \in U \text{ and } v \in V\}$$

- $\sim V = \Sigma^\omega - V$ is complement of $V \subseteq \Sigma^\omega$
- $U^\omega = \{u_0 \cdot u_1 \cdot \cdots \mid u_i \in U - \{\epsilon\} \text{ for all } i \in \mathbb{N}\}$ is ω–iteration of $U \subseteq \Sigma^*$

Definitions

- nondeterministic Büchi automaton (NBA) is NFA $M = (Q, \Sigma, \Delta, S, F)$ operating on Σ^ω
- run of M on input $x = a_0 a_1 a_2 \cdots \in \Sigma^\omega$ is infinite sequence $q_0, q_1, \dots$ of states such that $q_0 \in S$ and $q_{i+1} \in \Delta(q_i, a_i)$ for $i \geqslant 0$
- run $q_0, q_1, \dots$ is accepting if $q_i \in F$ for infinitely many i
- $L(M) = \{x \in \Sigma^\omega \mid x \text{ admits accepting run}\}$
- set $A \subseteq \Sigma^\omega$ is ω–regular if $A = L(M)$ for some NBA M
- deterministic Büchi automaton (DBA) is NBA $(Q, \Sigma, \Delta, S, F)$ with
  1. $|S| = 1$
  2. $|\Delta(q, a)| = 1$ for all $q \in Q$ and $a \in \Sigma$

Lemma

every ω–regular set is accepted by NBA with one start state

Theorem

not every ω–regular set is accepted by DBA

Theorem

1. ω–regular sets are effectively closed under union and intersection
2. left–concatenation of regular set and ω–regular set is ω–regular
3. ω–iteration of regular set is ω–regular

Theorem

set $A \subseteq \Sigma^\omega$ is ω–regular $\iff$ $A = U_1 \cdot V_1^\omega \cup \cdots \cup U_n \cdot V_n^\omega$ for some $n \in \mathbb{N}$ and regular $U_1, \ldots, U_n, V_1, \ldots, V_n \subseteq \Sigma^*$

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

Outline

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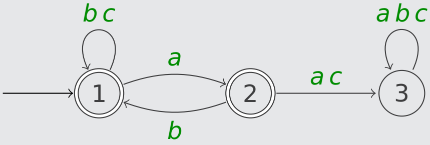
Example

DBA M



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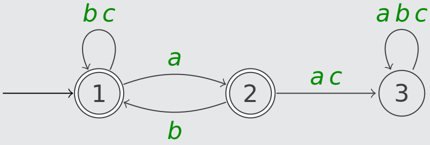

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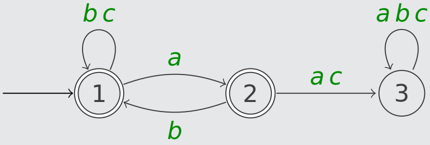

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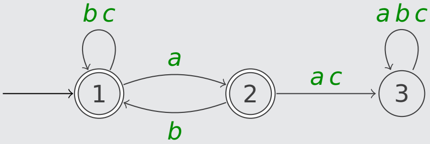

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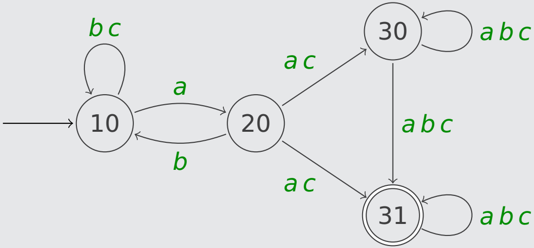

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Notation

for NBA $M = (Q, \Sigma, \Delta, S, F)$ and states $p, q \in Q$

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Definition

relation $\sim_M$ on Σ^* for NBA $M = (Q, \Sigma, \Delta, S, F)$: $u \sim_M v$ if for all $p, q \in Q$

$$u \in L_{pq} \iff v \in L_{pq}$$

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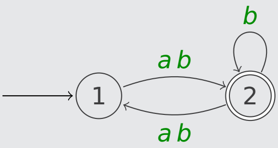
$$u \in L_{pq} \iff v \in L_{pq} \quad \text{and} \quad u \in L^{\mathrm{f}}_{pq} \iff v \in L^{\mathrm{f}}_{pq}$$

Example

NBA M



Example

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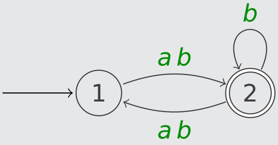

- $a \not\sim_M b$

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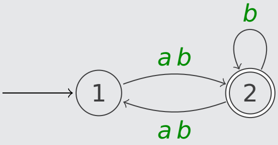

- $a \not\sim_M b$: $a \notin L_{22}$ and $b \in L_{22}$

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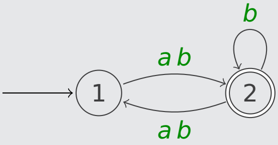

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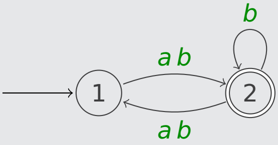

- $a \not\sim_M b$: $a \notin L_{22}$ and $b \in L_{22}$
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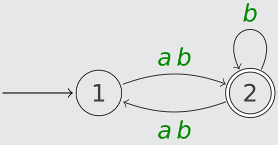

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- $a \sim_M aaa$

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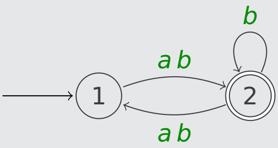

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for string $u \in \Sigma^*$

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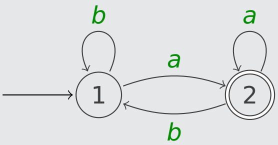
Example

- NBA M



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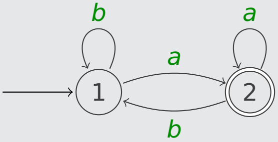

- $L_{11} = L((b + aa^*b)^*)$

Example

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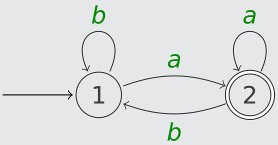

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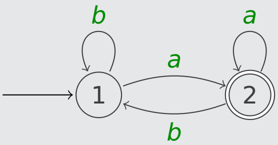

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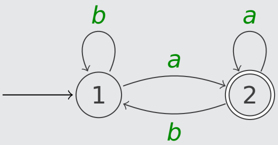

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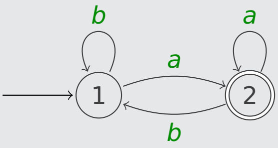

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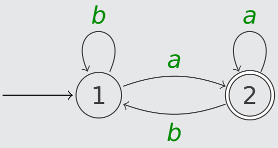

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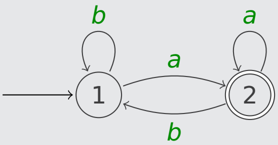

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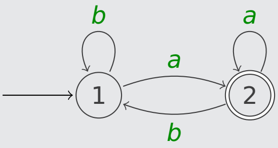

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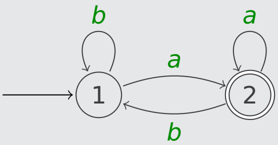

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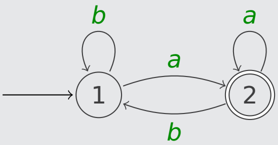

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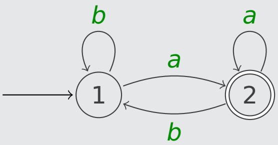

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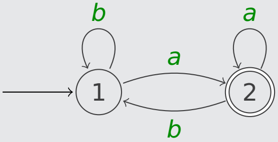
Example

- NBA M



- $L_{11} = L((a^*b)^*)$

  $L_{12} = L((a+b)^*a)$

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  $L_{22} = L((b^*a)^*)$

  $L^{\mathrm{f}}_{11} = L_{12} \cdot L_{21} = L((a+b)^*a(a+b)^*b)$

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Example

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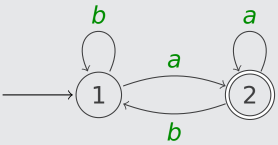

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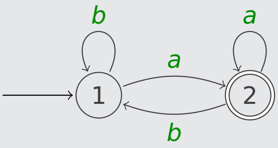
Example

- NBA M



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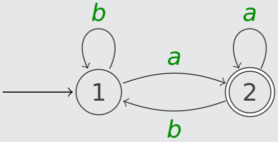

- $$\begin{aligned}
L_{11} &= L((a^*b)^*) & L^{\mathrm{f}}_{11} &= L_{12}\cdot L_{21} = L((a+b)^*a(a+b)^*b)\\
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L_{21} &= L((a+b)^*b) & L^{\mathrm{f}}_{21} &= L_{22}\cdot L_{21} = L((b^*a)^*(a+b)^*b) = L((a+b)^*b) = L_{21}\\
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\end{aligned}$$

Example

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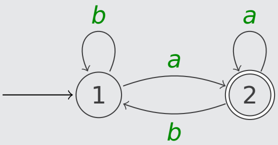

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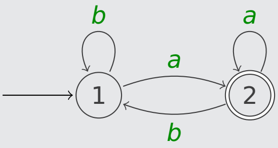
Example

- NBA M



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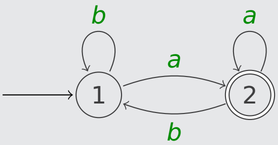
Example

- NBA M



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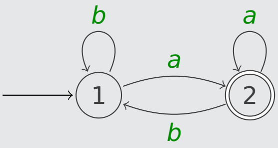
Example

- NBA M



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Example

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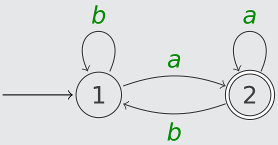

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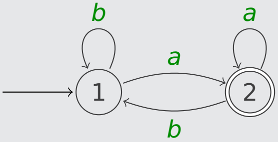
Example

- NBA M



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Example

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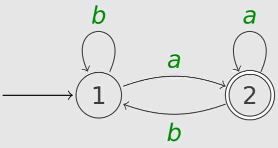

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Example

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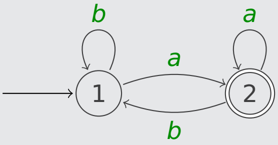

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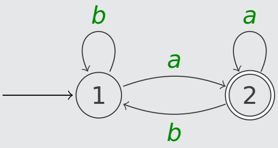

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Example

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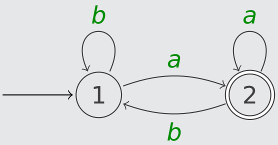

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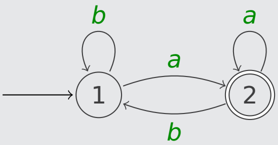
Example

- NBA M



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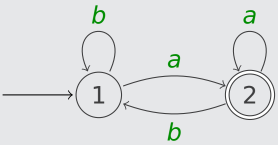

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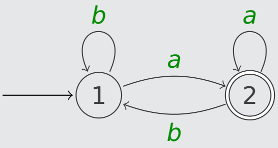

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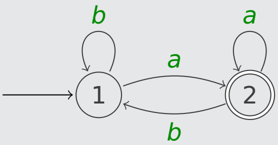

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- $[\epsilon]_{\sim_M} = \{\epsilon\}$

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Lemma

for all $x \in U \cdot V^\omega$ with equivalence classes U and V of $\sim_M$

- $x \in L(M) \implies U \cdot V^\omega \subseteq L(M)$

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Proof

- $x = uv_1v_2\cdots$ with $u \in U$ and $v_i \in V - \{\epsilon\}$ for all $i \geqslant 1$

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- $u \sim_M u'$ and $v_i \sim_M v_i'$ for all $i \geqslant 1 \implies x$ and x' have essentially same runs

Theorem (Ramsey)

every infinite k−colored complete graph contains infinite complete monochromatic subgraph

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  - w_{i+1} is arbitrary node in V_i and V_{i+1} is infinite subset of V_i such that all edges $\{w_{i+1} v \mid v \in V_{i+1}\}$ are same color
- $i < j < k \implies w_j, w_k \in V_i \implies$ edges $w_i w_j$ and $w_i w_k$ are same color
- w_i is blue-based (red-based) if edge $w_i w_j$ is blue (red) for all $j > i$

Theorem (Ramsey)

every infinite k–colored complete graph contains infinite complete monochromatic subgraph

Proof (for $k = 2$)

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Corollary

$$\sim L(M) = \bigcup \left\{ U \cdot V^{\omega} \;\middle|\; U \text{ and } V \text{ are equivalence classes of } \sim_M \text{ such that } U \cdot V^{\omega} \cap L(M) = \varnothing \right\}$$

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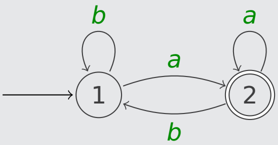
- optimal construction (Safra 1988) produces $2^{\mathcal{O}(n \log n)}$ states

Example

- NBA M



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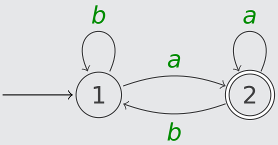

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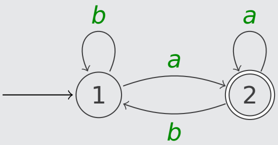

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	$①^\omega$	$②^\omega$	$③^\omega$	$④^\omega$
①				
②				
③				
④				

Example

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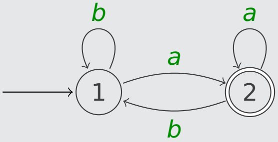

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$\textcircled{1}$				
$\textcircled{2}$				
$\textcircled{3}$				
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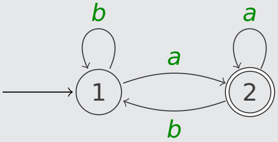

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	$❶^\omega$	$❷^\omega$	$❸^\omega$	$❹^\omega$
❶	🙂			
❷	🙂			
❸	🙂			
❹	🙂			

Example

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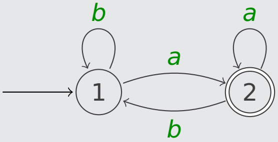

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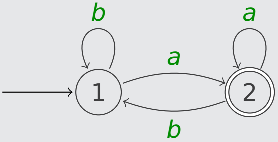
①	🙂	🙁		
②	🙂	🙁		
③	🙂	🙁		
④	🙂	🙁		

Example

- NBA M



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❶	🙂	🙁	🙂	
❷	🙂	🙁	🙂	
❸	🙂	🙁	🙂	
❹	🙂	🙁	🙂	

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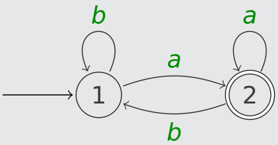

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①	🙂	🙁	🙂	🙁
②	🙂	🙁	🙂	🙁
③	🙂	🙁	🙂	🙁
④	🙂	🙁	🙂	🙁

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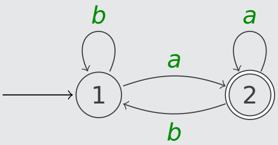

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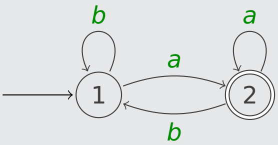

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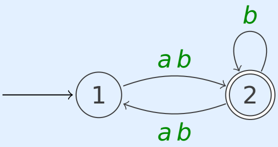
- $\sim L(M) = (① \cup ② \cup ③ \cup ④) \cdot (①^\omega \cup ③^\omega) = \{a, b\}^*\{b\}^\omega$

Outline

Question

What can be said about the equivalence relation $\sim_M$ for the following NBA M?



A $[\epsilon]_{\sim_M} = \{\epsilon\}$

B $ab \sim_M bb$

C $a \sim_M x$ for all $x \in L(a(aa)^*)$

D $ab \sim_M x$ for all $x \in L(ab^*)$

Outline

Definitions

- first–order variables $V_1 = \{x, y, \ldots\}$ ranging over natural numbers
- second–order variables $V_2 = \{X, Y, \ldots\}$ ranging over sets of natural numbers
- formulas of monadic second–order logic (MSO)

$$\varphi \ ::= \ \bot \mid x < y \mid X(x) \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists x.\,\varphi \mid \exists X.\,\varphi$$

with $x, y \in V_1$ and $X \in V_2$

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- assignment α is mapping from variables $x \in V_1$ to $\mathbb{N}$ and $X \in V_2$ to subsets of $\mathbb{N}$
- assignment α satisfies formula φ ($\alpha \vDash \varphi$):

$$\alpha \nvDash \bot$$

$$\begin{array}{lll@{\qquad}lll}
\alpha \vDash x < y & \iff & \alpha(x) < \alpha(y) & \alpha \vDash \neg\varphi & \iff & \alpha \nvDash \varphi \\
\alpha \vDash X(x) & \iff & \alpha(x) \in \alpha(X) & \alpha \vDash \varphi_1 \vee \varphi_2 & \iff & \alpha \vDash \varphi_1 \text{ or } \alpha \vDash \varphi_2 \\
\alpha \vDash \exists x.\, \varphi & \iff & \multicolumn{4}{l}{\alpha[x \mapsto n] \vDash \varphi \text{ for some } n \in \mathbb{N}} \\
\alpha \vDash \exists X.\, \varphi & \iff & \multicolumn{4}{l}{\alpha[X \mapsto N] \vDash \varphi \text{ for some subset } N \subseteq \mathbb{N}}
\end{array}$$

Example

formula $(\forall\, x.\, x = 0 \to X(x)) \land (\forall\, x.\, X(x) \to \exists\, y.\, x < y \land X(y))$

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- $L_a(\varphi) = \{x \in (\{0,1\}^{m+n})^\omega \mid x \text{ is } m\text{–admissible and } \underline{x} \vDash \varphi\}$

Example

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- $L_a(\varphi) = \binom{0}{1}^* \left[\binom{1}{0} + \binom{1}{1}\binom{0}{1}^*\binom{0}{0}\right]\left[\binom{0}{0} + \binom{0}{1}\right]^\omega$

Example

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- $L_a(\varphi) = \binom{0}{1}^* \left[\binom{1}{0} + \binom{1}{1}\binom{0}{1}^*\binom{0}{0}\right]\left[\binom{0}{0} + \binom{0}{1}\right]^\omega$

Theorem

$L_a(\varphi)$ is ω–regular for every MSO formula φ

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Theorem

set $A \subseteq \Sigma^\omega$ is ω–regular if and only if A is MSO definable

Examples

- $\{x \in \{a, b\}^\omega \mid |x|_a = \infty\}$ is represented by MSO formula

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$$\exists\, x.\, \forall\, y.\, x < y \to P_b(y)$$

Outline

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- monadic second–order logic
- MSO
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homework for December 12