



Automata and Logic

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Outline

- 1. Summary of Previous Lecture**
- 2. Complementation**
- 3. Intermezzo**
- 4. Monadic Second-Order Logic**
- 5. Further Reading**

Definitions

- ▶ **infinite string** over alphabet Σ is function $x: \mathbb{N} \rightarrow \Sigma$
- ▶ Σ^ω denotes set of all infinite strings over Σ
- ▶ $|x|_a$ for $x \in \Sigma^\omega$ and $a \in \Sigma$ denotes number of occurrences of a in x
- ▶ **left-concatenation** of $u \in \Sigma^*$ and $v \in \Sigma^\omega$ is denoted by $u \cdot v \in \Sigma^\omega$
- ▶ **left-concatenation** of $U \subseteq \Sigma^*$ and $V \subseteq \Sigma^\omega$

$$U \cdot V = \{u \cdot v \mid u \in U \text{ and } v \in V\}$$

- ▶ $\sim V = \Sigma^\omega - V$ is **complement** of $V \subseteq \Sigma^\omega$
- ▶ $U^\omega = \{u_0 \cdot u_1 \cdot \dots \mid u_i \in U - \{\epsilon\} \text{ for all } i \in \mathbb{N}\}$ is **ω -iteration** of $U \subseteq \Sigma^*$

Definitions

- ▶ **nondeterministic Büchi automaton (NBA)** is NFA $M = (Q, \Sigma, \Delta, S, F)$ operating on Σ^ω
- ▶ **run** of M on input $x = a_0a_1a_2\cdots \in \Sigma^\omega$ is infinite sequence $q_0, q_1, \dots$ of states such that $q_0 \in S$ and $q_{i+1} \in \Delta(q_i, a_i)$ for $i \geq 0$
- ▶ run $q_0, q_1, \dots$ is **accepting** if $q_i \in F$ for infinitely many i
- ▶ $L(M) = \{x \in \Sigma^\omega \mid x \text{ admits accepting run}\}$
- ▶ set $A \subseteq \Sigma^\omega$ is **ω -regular** if $A = L(M)$ for some NBA M
- ▶ **deterministic Büchi automaton (DBA)** is NBA $(Q, \Sigma, \Delta, S, F)$ with
 - ① $|S| = 1$
 - ② $|\Delta(q, a)| = 1$ for all $q \in Q$ and $a \in \Sigma$

Lemma

every ω -regular set is accepted by NBA with one start state

Theorem

not every ω -regular set is accepted by DBA

Theorem

- 1 ω -regular sets are effectively closed under union and intersection
- 2 left-concatenation of regular set and ω -regular set is ω -regular
- 3 ω -iteration of regular set is ω -regular

Theorem

set $A \subseteq \Sigma^\omega$ is ω -regular $\iff A = U_1 \cdot V_1^\omega \cup \dots \cup U_n \cdot V_n^\omega$
for some $n \in \mathbb{N}$ and regular $U_1, \dots, U_n, V_1, \dots, V_n \subseteq \Sigma^*$

Automata

- ▶ (deterministic, nondeterministic, alternating) finite automata
- ▶ regular expressions
- ▶ (alternating) Büchi automata

Logic

- ▶ (weak) monadic second-order logic
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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Theorem

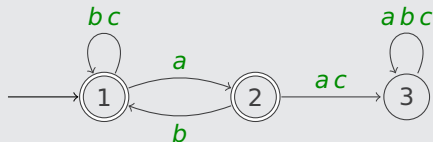
every DBA M can be effectively transformed into NBA M' such that $L(M') = \sim L(M)$

Proof

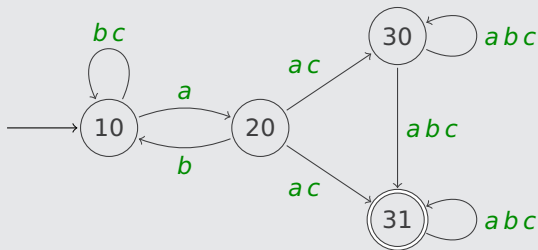
- ▶ DBA $M = (Q, \Sigma, \Delta, S, F)$
- ▶ NBA M' should accept all strings for which (single) run of M visits F finitely often
idea: guess state from which no more visits to F take place
- ▶ $Q' = (Q \times \{0\}) \cup ((Q - F) \times \{1\})$
- ▶ $S' = S \times \{0\}$
- ▶ $F' = (Q - F) \times \{1\}$
- ▶ if $\Delta(p, a) = \{q\}$ then $\Delta'((p, i), a) = \begin{cases} \{(q, 0), (q, 1)\} & \text{if } i = 0 \text{ and } q \notin F \\ \{(q, 0)\} & \text{if } i = 0 \text{ and } q \in F \\ \{(q, 1)\} & \text{if } i = 1 \text{ and } q \notin F \\ \emptyset & \text{if } i = 1 \text{ and } q \in F \end{cases}$

Example

DBA M



- ▶ $L(M) = \{x \in \{a, b, c\}^\omega \mid \text{every } a \text{ in } x \text{ is immediately followed by } b\}$
- ▶ $\sim L(M) = \{x \in \{a, b, c\}^\omega \mid x \text{ contains } aa \text{ or } ac \text{ as substring}\}$ is accepted by NBA M'



Theorem

ω -regular sets are closed under complement

Notation

for NBA $M = (Q, \Sigma, \Delta, S, F)$ and states $p, q \in Q$

$$\triangleright L_{pq} = \{x \in \Sigma^* \mid q \in \hat{\Delta}(\{p\}, x)\}$$

$$\triangleright L_{pq}^f = \bigcup_{f \in F} L_{pf} \cdot L_{fq}$$

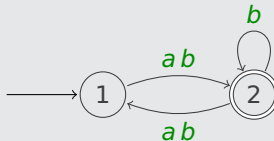
Definition

relation $\sim_M$ on Σ^* for NBA $M = (Q, \Sigma, \Delta, S, F)$: $u \sim_M v$ if for all $p, q \in Q$

$$u \in L_{pq} \iff v \in L_{pq} \quad \text{and} \quad u \in L_{pq}^f \iff v \in L_{pq}^f$$

Example

NBA M



- ▶ $a \not\sim_M b$: $a \notin L_{22}$ and $b \in L_{22}$
- ▶ $b \not\sim_M bb$: $b \notin L_{11}$ and $bb \in L_{11}$
- ▶ $a \sim_M aaa$
- ▶ $bb \sim_M bbb$

Lemma

- ① $\sim_M$ is equivalence relation of finite index
- ② each equivalence class of $\sim_M$ is regular

Proof

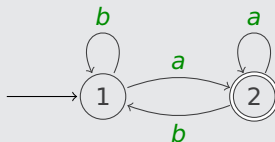
for string $u \in \Sigma^*$

- ▶ $Q_u = \{(p, q) \in Q \times Q \mid u \in L_{pq}\}$
- ▶ $Q_u^f = \{(p, q) \in Q \times Q \mid u \in L_{pq}^f\}$
- ▶ $Q_{\neg u} = Q \times Q - Q_u$
- ▶ $Q_{\neg u}^f = Q \times Q - Q_u^f$

$$[u]_{\sim_M} = \left(\bigcap_{(p,q) \in Q_u} L_{pq} \cap \bigcap_{(p,q) \in Q_u^f} L_{pq}^f \right) - \left(\bigcup_{(p,q) \in Q_{\neg u}} L_{pq} \cup \bigcup_{(p,q) \in Q_{\neg u}^f} L_{pq}^f \right)$$

Example

► NBA M



$$\text{► } L_{11} = L((b + aa^*b)^*) = L((a^*b)^*)$$

$$L_{11}^f = L((a + b)^* a (a + b)^* b)$$

$$L_{12} = L((a + b)^* a)$$

$$L_{12}^f = L_{12}$$

$$L_{21} = L((a + b)^* b)$$

$$L_{21}^f = L_{21}$$

$$L_{22} = L((b^*a)^*)$$

$$L_{22}^f = L_{22}$$

$$\text{► } [\epsilon]_{\sim_M} = \{\epsilon\}$$

$$[a]_{\sim_M} = L((a + b)^* a)$$

$$[\epsilon]_{\sim_M} \cup [a]_{\sim_M} \cup [b]_{\sim_M} \cup [ab]_{\sim_M} = \{a, b\}^*$$

$$[b]_{\sim_M} = L(b^+)$$

$$[ab]_{\sim_M} = L((a + b)^* a (a + b)^* b)$$

Lemma

for all $x \in U \cdot V^\omega$ with equivalence classes U and V of $\sim_M$

- ▶ $x \in L(M) \implies U \cdot V^\omega \subseteq L(M)$
- ▶ $x \notin L(M) \implies U \cdot V^\omega \subseteq \sim L(M)$

Proof

- ▶ $x = uv_1v_2\cdots$ with $u \in U$ and $v_i \in V - \{\epsilon\}$ for all $i \geq 1$
- ▶ arbitrary $x' \in U \cdot V^\omega$ can be written as $u'v'_1v'_2\cdots$ with $u' \in U$ and $v'_i \in V - \{\epsilon\}$ for all $i \geq 1$
- ▶ $u \sim_M u'$ and $v_i \sim_M v'_i$ for all $i \geq 1 \implies x$ and x' have essentially same runs

Theorem (Ramsey)

every infinite k -colored **complete** graph contains infinite complete monochromatic subgraph

Proof (for $k = 2$)

- ▶ $V = \{v_i \mid i \in \mathbb{N}\}$ is arbitrary countable subset of nodes of given graph; 2 colors: **red** **blue**
- ▶ construct infinite sequences $w_0, w_1, \dots \in V$ and $V_0, V_1, \dots \subseteq V$ inductively:
 - ▶ $w_0 = v_0$ and V_0 is infinite subset of V such that all edges $\{w_0 v \mid v \in V_0\}$ are same color
 - ▶ w_{i+1} is arbitrary node in V_i and V_{i+1} is infinite subset of V_i such that all edges $\{w_{i+1} v \mid v \in V_{i+1}\}$ are same color
- ▶ $i < j < k \implies w_j, w_k \in V_i \implies$ edges $w_i w_j$ and $w_i w_k$ are same color
- ▶ w_i is **blue**-based (**red**-based) if edge $w_i w_j$ is **blue** (**red**) for all $j > i$
- ▶ pigeon-hole principle: $\exists w_{i_0}, w_{i_1}, \dots$ such that all elements have same color base
- ▶ $w_{i_0}, w_{i_1}, \dots$ induces infinite complete monochromatic subgraph

Lemma

for all $x \in \Sigma^\omega$ there exist equivalence classes U and V of $\sim_M$ such that $x \in U \cdot V^\omega$

Proof

- ▶ $x = a_0 a_1 a_2 \dots$
- ▶ for all $i < j$ "color" c_{ij} is equivalence class of $\sim_M$ containing finite substring $x_{ij} = a_i \dots a_j$
- ▶ $\sim_M$ is of finite index $\implies$ (theorem of Ramsey)
 $\exists$ equivalence class V and infinite subset S of $\mathbb{N}$ such that $x_{ij} \in V$ for all $i, j \in S$ with $i < j$
- ▶ $U = [a_0 \dots a_{k-1}]_{\sim_M}$ where $k = \min(S)$
- ▶ $x \in U \cdot V^\omega$

Corollary

$\sim L(M) = \bigcup \{ U \cdot V^\omega \mid U \text{ and } V \text{ are equivalence classes of } \sim_M \text{ such that } U \cdot V^\omega \cap L(M) = \emptyset \}$

Corollary

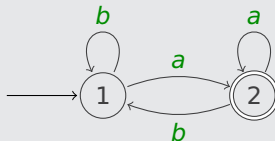
$\sim L(M)$ is ω -regular for every NBA M

Remarks

- ▶ construction is double exponential:
if M has n states then NBA for $\sim L(M)$ has $2^{2^{\mathcal{O}(n)}}$ states
- ▶ optimal construction (Safra 1988) produces $2^{\mathcal{O}(n \log n)}$ states

Example

► NBA M



► equivalence classes of $\sim_M$:

$$\textcircled{1} = \{\epsilon\} \quad \textcircled{2} = L((a+b)^*a) \quad \textcircled{3} = L(b^+) \quad \textcircled{4} = L((a+b)^*a(a+b)^*b)$$

► $U \cdot V^\omega \cap L(M) = \emptyset$ for $U, V \in \{\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}\}$?

	$\textcircled{1}^\omega$	$\textcircled{2}^\omega$	$\textcircled{3}^\omega$	$\textcircled{4}^\omega$
$\textcircled{1}$	😊	😞	😊	😞
$\textcircled{2}$	😊	😞	😊	😞
$\textcircled{3}$	😊	😞	😊	😞
$\textcircled{4}$	😊	😞	😊	😞

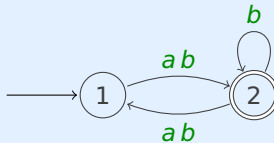
$$\text{► } \sim L(M) = (\textcircled{1} \cup \textcircled{2} \cup \textcircled{3} \cup \textcircled{4}) \cdot (\textcircled{1}^\omega \cup \textcircled{3}^\omega) = \{a, b\}^* \{b\}^\omega$$

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Question

What can be said about the equivalence relation $\sim_M$ for the following NBA M ?



- A $[\epsilon]_{\sim_M} = \{\epsilon\}$
- B $ab \sim_M bb$
- C $a \sim_M x$ for all $x \in L(a(aa)^*)$
- D $ab \sim_M x$ for all $x \in L(ab^*)$



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Definitions

- ▶ first-order variables $V_1 = \{x, y, \dots\}$ ranging over natural numbers
- ▶ second-order variables $V_2 = \{X, Y, \dots\}$ ranging over **sets of natural numbers**
- ▶ formulas of **monadic second-order logic (MSO)**

$$\varphi ::= \perp \mid x < y \mid X(x) \mid \neg \varphi \mid \varphi_1 \vee \varphi_2 \mid \exists x. \varphi \mid \exists X. \varphi$$

with $x, y \in V_1$ and $X \in V_2$

- ▶ assignment α is mapping from variables $x \in V_1$ to $\mathbb{N}$ and $X \in V_2$ to **subsets of $\mathbb{N}$**
- ▶ assignment α satisfies formula φ ($\alpha \models \varphi$):

$$\alpha \not\models \perp$$

$$\alpha \models x < y \iff \alpha(x) < \alpha(y) \qquad \alpha \models \neg \varphi \iff \alpha \not\models \varphi$$

$$\alpha \models X(x) \iff \alpha(x) \in \alpha(X) \qquad \alpha \models \varphi_1 \vee \varphi_2 \iff \alpha \models \varphi_1 \text{ or } \alpha \models \varphi_2$$

$$\alpha \models \exists x. \varphi \iff \alpha[x \mapsto n] \models \varphi \text{ for some } n \in \mathbb{N}$$

$$\alpha \models \exists X. \varphi \iff \alpha[X \mapsto N] \models \varphi \text{ for some subset } N \subseteq \mathbb{N}$$

Example

formula $(\forall x. x = 0 \rightarrow X(x)) \wedge (\forall x. X(x) \rightarrow \exists y. x < y \wedge X(y))$

- ▶ satisfiable: $\alpha(X)$ is infinite subset of $\mathbb{N}$ containing 0
- ▶ not satisfiable in WMSO

Definitions

- ▶ assignment α for MSO formula φ with $FV(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is encoded as infinite string $\underline{\alpha} \in (\{0, 1\}^{m+n})^\omega$:

$$\alpha(x_i) = j \quad \text{if } i\text{-th entry of } j\text{-th symbol in } \underline{\alpha} \text{ is } 1$$

$$\alpha(X_i) = \{j \mid (m+i)\text{-th entry of } j\text{-th symbol in } \underline{\alpha} \text{ is } 1\}$$

- ▶ infinite string over $\{0, 1\}^{m+n}$ is **m -admissible** if first m rows contain exactly one 1 each
- ▶ $L_a(\varphi) = \{x \in (\{0, 1\}^{m+n})^\omega \mid x \text{ is } m\text{-admissible and } \underline{x} \models \varphi\}$

Example

- ▶ $\varphi(x, X) = (\forall y. y < x \rightarrow X(y)) \wedge (\exists y. \neg X(y))$
- ▶ $L_a(\varphi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^\omega$

Theorem

$L_a(\varphi)$ is ω -regular for every MSO formula φ

Remark

proof similar to WMSO case, using (closure properties of) NBAs

Theorem

set $A \subseteq \Sigma^\omega$ is ω -regular if and only if A is MSO definable

Examples

- ▶ $\{x \in \{a, b\}^\omega \mid |x|_a = \infty\}$ is represented by MSO formula

$$\forall x. \exists y. x < y \wedge P_a(y)$$

- ▶ $\{x \in \{a, b\}^\omega \mid |x|_a \neq \infty\}$ is represented by MSO formula

$$\exists x. \forall y. x < y \rightarrow P_b(y)$$

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Important Concepts

- ▶ $\sim_M$
- ▶ monadic second-order logic
- ▶ MSO
- ▶ Ramsey's theorem

homework for December 5

homework for December 12