universität innsbruck

25W lecture 11

Automata and Logic

Aart Middeldorp and Samuel Frontull

Outline

Theorem

every DBA M can be effectively transformed into NBA M' such that $L(M') = \sim L(M)$

Theorem

ω–regular sets are closed under complement

Notation

for NBA $M = (Q, \Sigma, \Delta, S, F)$ and states $p, q \in Q$

$$L_{pq} = \{x \in \Sigma^* \mid q \in \widehat{\Delta}(\{p\}, x)\} \qquad\qquad L^{\mathrm{f}}_{pq} = \bigcup_{f \in F} L_{pf} \cdot L_{fq}$$

Definition

relation $\sim_M$ on Σ^* for NBA $M = (Q, \Sigma, \Delta, S, F)$: $u \sim_M v$ if for all $p, q \in Q$

$$u \in L_{pq} \iff v \in L_{pq} \quad \text{and} \quad u \in L^{\mathrm{f}}_{pq} \iff v \in L^{\mathrm{f}}_{pq}$$

Lemma

1. $\sim_M$ is equivalence relation of finite index
2. each equivalence class of $\sim_M$ is regular

Lemma

for all $x \in U \cdot V^\omega$ with equivalence classes U and V of $\sim_M$

$$x \in L(M) \implies U \cdot V^\omega \subseteq L(M) \qquad\qquad x \notin L(M) \implies U \cdot V^\omega \subseteq \sim L(M)$$

Lemma

for all $x \in \Sigma^\omega$ there exist equivalence classes U and V of $\sim_M$ such that $x \in U \cdot V^\omega$

Corollary

$\sim L(M) = \bigcup \left\{ U \cdot V^\omega \mid U \text{ and } V \text{ are equivalence classes of } \sim_M \text{ such that } U \cdot V^\omega \cap L(M) = \varnothing \right\}$

Definitions

- first–order variables $V_1 = \{x, y, \ldots\}$ ranging over natural numbers
- second–order variables $V_2 = \{X, Y, \ldots\}$ ranging over sets of natural numbers
- formulas of monadic second–order logic (MSO)

$$\varphi \ ::= \ \bot \mid x < y \mid X(x) \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists\, x.\, \varphi \mid \exists\, X.\, \varphi$$

  with $x, y \in V_1$ and $X \in V_2$
- assignment α is mapping from variables $x \in V_1$ to $\mathbb{N}$ and $X \in V_2$ to subsets of $\mathbb{N}$
- assignment α satisfies formula φ $(\alpha \vDash \varphi)$:

$$\begin{array}{lcllcl}
\alpha \nvDash \bot & & & & & \\
\alpha \vDash x < y & \iff & \alpha(x) < \alpha(y) & \alpha \vDash \neg\varphi & \iff & \alpha \nvDash \varphi \\
\alpha \vDash X(x) & \iff & \alpha(x) \in \alpha(X) & \alpha \vDash \varphi_1 \vee \varphi_2 & \iff & \alpha \vDash \varphi_1 \text{ or } \alpha \vDash \varphi_2 \\
\alpha \vDash \exists\, x.\, \varphi & \iff & \alpha[x \mapsto n] \vDash \varphi \text{ for some } n \in \mathbb{N} & & & \\
\alpha \vDash \exists\, X.\, \varphi & \iff & \alpha[X \mapsto N] \vDash \varphi \text{ for some subset } N \subseteq \mathbb{N} & & &
\end{array}$$

Definitions

- assignment α for MSO formula φ with $\mathrm{FV}(\varphi) = (x_1, \dots, x_m, X_1, \dots, X_n)$ is encoded as infinite string $\underline{\alpha} \in (\{0,1\}^{m+n})^\omega$:

$$\alpha(x_i) = j \qquad \text{if } i\text{–th entry of } j\text{–th symbol in } \underline{\alpha} \text{ is } 1$$

$$\alpha(X_i) = \{\, j \mid (m+i)\text{–th entry of } j\text{–th symbol in } \underline{\alpha} \text{ is } 1 \,\}$$

- infinite string over $\{0,1\}^{m+n}$ is m–admissible if first m rows contain exactly one 1 each
- $L_a(\varphi) = \{x \in (\{0,1\}^{m+n})^\omega \mid x \text{ is } m\text{–admissible and } \underline{x} \vDash \varphi\}$

Theorem

$L_a(\varphi)$ is ω–regular for every MSO formula φ

Theorem

set $A \subseteq \Sigma^\omega$ is ω–regular if and only if A is MSO definable

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

Outline

Remark

minimal DBAs are not unique

Remark

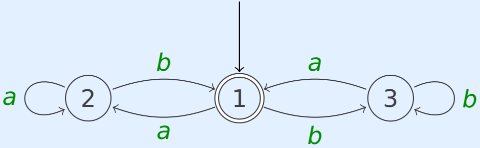
minimal DBAs are not unique

- $L = \{x \in \{a, b\}^\omega \mid |x|_a = \infty \text{ and } |x|_b = \infty\}$

Remark

minimal DBAs are not unique

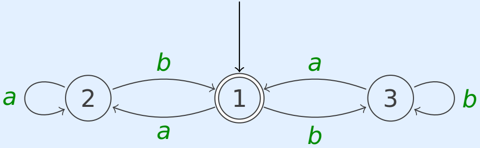
- $L = \{x \in \{a, b\}^\omega \mid |x|_a = \infty$ and $|x|_b = \infty\}$
- $L = L(M_1)$ for DBA M_1



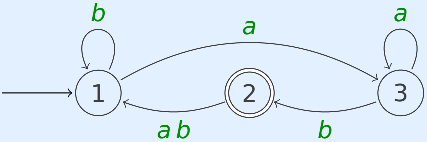
Remark

minimal DBAs are not unique

- $L = \{x \in \{a, b\}^\omega \mid |x|_a = \infty$ and $|x|_b = \infty\}$
- $L = L(M_1)$ for DBA M_1



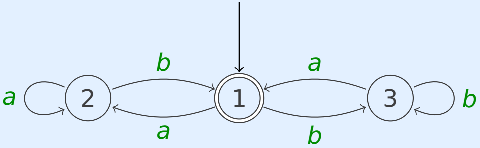
- $L = L(M_2)$ for DBA M_2



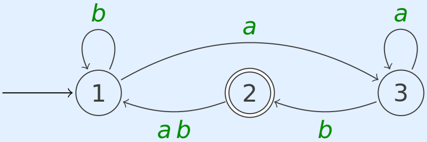
Remark

minimal DBAs are not unique

- $L = \{x \in \{a, b\}^\omega \mid |x|_a = \infty \text{ and } |x|_b = \infty\}$
- $L = L(M_1)$ for DBA M_1



- $L = L(M_2)$ for DBA M_2

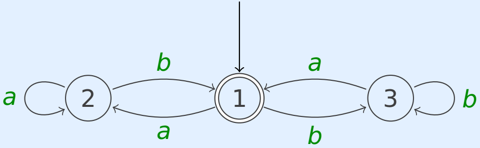


- M_1 and M_2 are not isomorphic

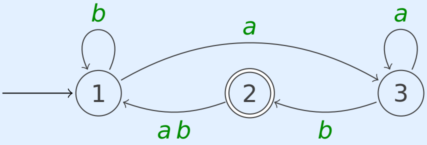
Remark

minimal DBAs are not unique

- $L = \{x \in \{a, b\}^\omega \mid |x|_a = \infty \text{ and } |x|_b = \infty\}$
- $L = L(M_1)$ for DBA M_1



- $L = L(M_2)$ for DBA M_2



- M_1 and M_2 are not isomorphic
- no DBA with 2 states accepts L

Definition

- generalized Büchi automaton (GBA) is automaton $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$ with $F_1, \ldots, F_k \subseteq Q$

Definition

- generalized Büchi automaton (GBA) is automaton $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$ with $F_1, \ldots, F_k \subseteq Q$
- $\infty(r) = \{q \in Q \mid q \text{ occurs infinitely often in run } r\}$

Definition

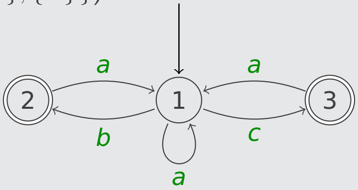
- generalized Büchi automaton (GBA) is automaton $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$ with $F_1, \ldots, F_k \subseteq Q$
- $\infty(r) = \{q \in Q \mid q \text{ occurs infinitely often in run } r\}$
- run r of GBA is accepting if $\infty(r) \cap F_i \neq \varnothing$ for all $1 \leqslant i \leqslant k$

Definition

- generalized Büchi automaton (GBA) is automaton $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$ with $F_1, \dots, F_k \subseteq Q$
- $\infty(r) = \{q \in Q \mid q \text{ occurs infinitely often in run } r\}$
- run r of GBA is accepting if $\infty(r) \cap F_i \neq \varnothing$ for all $1 \leqslant i \leqslant k$

Example

GBA $M = (\{1, 2, 3\}, \{a, b, c\}, \Delta, \{1\}, \{\{2\}, \{3\}\})$

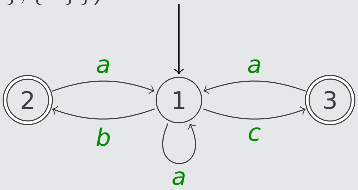


Definition

- generalized Büchi automaton (GBA) is automaton $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$ with $F_1, \dots, F_k \subseteq Q$
- $\infty(r) = \{q \in Q \mid q \text{ occurs infinitely often in run } r\}$
- run r of GBA is accepting if $\infty(r) \cap F_i \neq \varnothing$ for all $1 \leqslant i \leqslant k$

Example

GBA $M = (\{1,2,3\}, \{a,b,c\}, \Delta, \{1\}, \{\{2\},\{3\}\})$



$L(M) = \{x \in \{a,b,c\}^\omega \mid |x|_b = |x|_c = \infty \text{ and each } b \text{ and } c \text{ in } x \text{ is followed by } a\}$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Proof

- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Proof

- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$
- $L(M) = L(M')$ for NBA $M' = (Q', \Sigma, \Delta', S', F')$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Proof

- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$
- $L(M) = L(M')$ for NBA $M' = (Q', \Sigma, \Delta', S', F')$ with
  - $Q' = Q \times \{1, \dots, k\}$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Proof

- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$
- $L(M) = L(M')$ for NBA $M' = (Q', \Sigma, \Delta', S', F')$ with
  - $Q' = Q \times \{1, \ldots, k\}$
  - $S' = S \times \{1\}$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Proof

- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$
- $L(M) = L(M')$ for NBA $M' = (Q', \Sigma, \Delta', S', F')$ with
  - $Q' = Q \times \{1, \dots, k\}$
  - $S' = S \times \{1\}$
  - $\Delta'((q,i),a) = \Delta(q,a) \times \{j\}$ with $j = \begin{cases} i & \text{if } q \notin F_i \\ & \end{cases}$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Proof

- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$
- $L(M) = L(M')$ for NBA $M' = (Q', \Sigma, \Delta', S', F')$ with
  - $Q' = Q \times \{1, \ldots, k\}$
  - $S' = S \times \{1\}$
  - $\Delta'((q,i),a) = \Delta(q,a) \times \{j\}$ with $j = \begin{cases} i & \text{if } q \notin F_i \\ (i \bmod k) + 1 & \text{if } q \in F_i \end{cases}$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

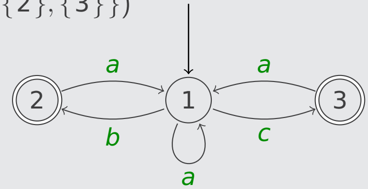
every GBA can be transformed into equivalent NBA

Proof

- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$
- $L(M) = L(M')$ for NBA $M' = (Q', \Sigma, \Delta', S', F')$ with
  - $Q' = Q \times \{1, \dots, k\}$
  - $S' = S \times \{1\}$
  - $\Delta'((q,i),a) = \Delta(q,a) \times \{j\}$ with $j = \begin{cases} i & \text{if } q \notin F_i \\ (i \bmod k) + 1 & \text{if } q \in F_i \end{cases}$
  - $F' = F_1 \times \{1\}$

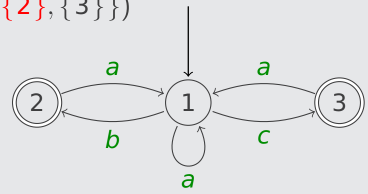
Example

GBA $M = (\{1,2,3\},\{a,b,c\},\Delta,\{1\},\{\{2\},\{3\}\})$

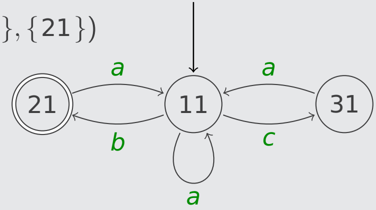


Example

GBA $M = (\{1, 2, 3\}, \{a, b, c\}, \Delta, \{1\}, \{\{2\}, \{3\}\})$

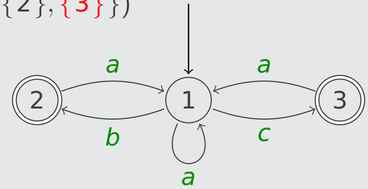


NBA $M' = (\{11, 21, 31, 12, 22, 32\}, \{a, b, c\}, \Delta', \{11\}, \{21\})$

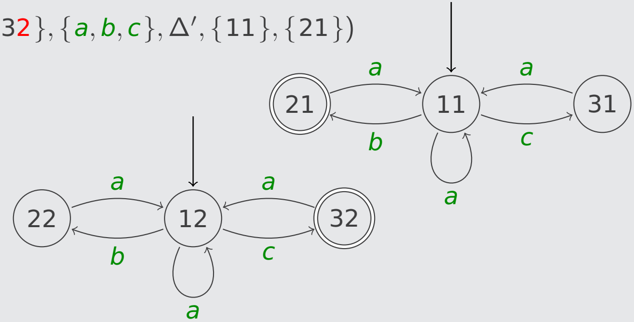


Example

GBA $M = (\{1,2,3\}, \{a,b,c\}, \Delta, \{1\}, \{\{2\}, \{3\}\})$

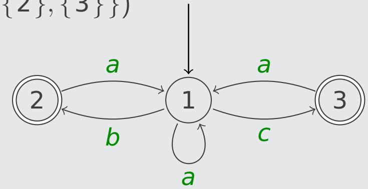


NBA $M' = (\{11,21,31,12,22,32\}, \{a,b,c\}, \Delta', \{11\}, \{21\})$

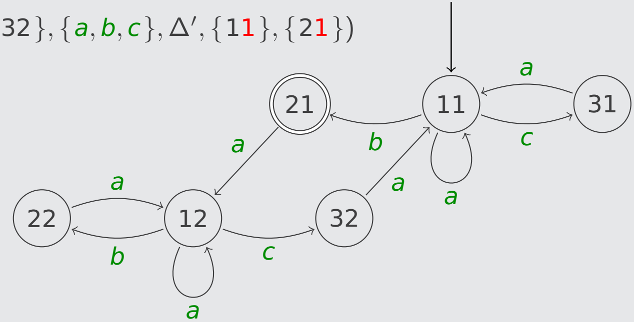


Example

GBA $M = (\{1,2,3\},\{a,b,c\},\Delta,\{1\},\{\{2\},\{3\}\})$



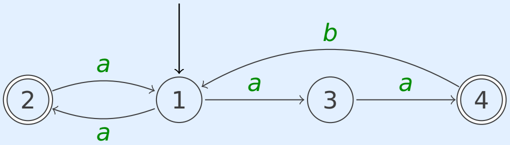
NBA $M' = (\{11,21,31,12,22,32\},\{a,b,c\},\Delta',\{11\},\{21\})$



Outline

Question

Which statements about the following GBA $M = (\{1,2,3,4\},\{a,b\},\Delta,\{1\},F)$ are true ?



A $L(M) = \{aa, aab\}^\omega$ if $F = \{\{2\},\{3\}\}$

B $L(M) = \{aa, aab\}^\omega$ if $F = \{\{2,3\}\}$

C it does not matter whether state 3 or 4 is final

D there is only one possible choice for F such that $L(M) = \{aa, aab\}^\omega$

Outline

Formal Verification

Formal Verification comprises

- framework for modeling systems (description language)

Formal Verification comprises

- framework for modeling systems (description language)
- specification language for describing properties to be verified

Formal Verification comprises

- framework for modeling systems (description language)
- specification language for describing properties to be verified
- verification method to establish whether description of system satisfies specification

Formal Verification comprises

- framework for modeling systems (description language)
- specification language for describing properties to be verified
- verification method to establish whether description of system satisfies specification

Model Checking

automatic formal verification approach for concurrent systems based on temporal logic

Formal Verification comprises

- framework for modeling systems (description language)
- specification language for describing properties to be verified
- verification method to establish whether description of system satisfies specification

Model Checking

automatic formal verification approach for concurrent systems based on temporal logic

Temporal Logic

- formulas are not statically true or false in model

Formal Verification comprises

- framework for modeling systems (description language)
- specification language for describing properties to be verified
- verification method to establish whether description of system satisfies specification

Model Checking

automatic formal verification approach for concurrent systems based on temporal logic

Temporal Logic

- formulas are not statically true or false in model
- models of temporal logic contain several states and truth is dynamic

Formal Verification comprises

- framework for modeling systems (description language)
- specification language for describing properties to be verified
- verification method to establish whether description of system satisfies specification

Model Checking

automatic formal verification approach for concurrent systems based on temporal logic

Temporal Logic

- formulas are not statically true or false in model
- models of temporal logic contain several states and truth is dynamic
- formula can be true in some states and false in others

Model Checking

- models are transition systems $\mathcal{M}$
- properties are formulas φ in temporal logic
- model checker determines whether $\mathcal{M} \vDash \varphi$ is true or not

Model Checking

- models are transition systems $\mathcal{M}$
- properties are formulas φ in temporal logic
- model checker determines whether $\mathcal{M} \vDash \varphi$ is true or not

Two Temporal Logics

- computation tree logic (CTL)
- linear–time temporal logic (LTL)

Model Checking

- models are transition systems $\mathcal{M}$
- properties are formulas φ in temporal logic
- model checker determines whether $\mathcal{M} \vDash \varphi$ is true or not

Two Temporal Logics

- computation tree logic (CTL)
- linear–time temporal logic (LTL)

Impact

both logics have been proven to be extremely fruitful in verifying hardware and communication protocols

Model Checking

- models are transition systems $\mathcal{M}$
- properties are formulas φ in temporal logic
- model checker determines whether $\mathcal{M} \vDash \varphi$ is true or not

Two Temporal Logics

- computation tree logic (CTL)
- linear–time temporal logic (LTL)

Impact

both logics have been proven to be extremely fruitful in verifying hardware and communication protocols, and are increasingly applied to software verification

Model Checking

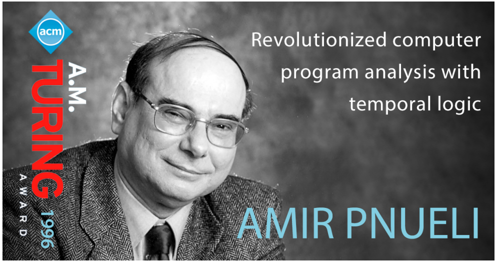
- models are transition systems $\mathcal{M}$
- properties are formulas φ in temporal logic
- model checker determines whether $\mathcal{M} \vDash \varphi$ is true or not

Two Temporal Logics

- computation tree logic (CTL)
- linear–time temporal logic (LTL)

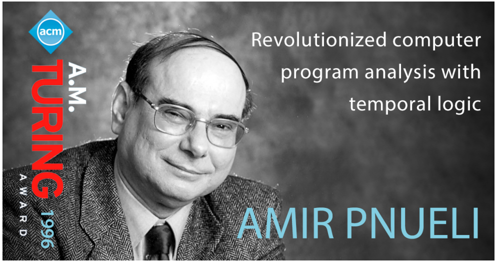
Impact

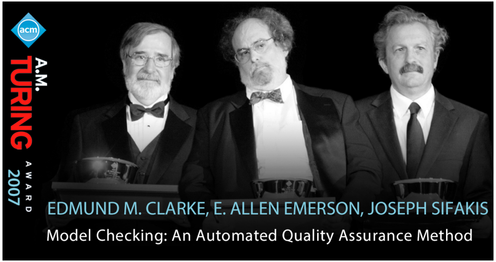
both logics have been proven to be extremely fruitful in verifying hardware and communication protocols, and are increasingly applied to software verification



ACM Turing Awards

1996 Amir Pnueli





ACM Turing Awards

1996 Amir Pnueli

2007 Edmund M. Clarke, E. Allen Emerson, Joseph Sifakis

Outline

Definitions

- LTL (linear–time temporal logic) formulas are built from
  - atoms $p, q, r, p_1, p_2, \ldots$
  - logical connectives $\bot, \top, \neg, \wedge, \vee, \rightarrow$

Definitions

- LTL (linear–time temporal logic) formulas are built from
  - atoms $p, q, r, p_1, p_2, \ldots$
  - logical connectives $\bot, \top, \neg, \wedge, \vee, \rightarrow$
  - temporal connectives $\mathsf{X}, \mathsf{F}, \mathsf{G}, \mathsf{U}, \mathsf{W}, \mathsf{R}$

Definitions

- LTL (linear–time temporal logic) formulas are built from
  - atoms $p, q, r, p_1, p_2, \ldots$
  - logical connectives $\bot, \top, \neg, \wedge, \vee, \rightarrow$
  - temporal connectives $\mathsf{X}, \mathsf{F}, \mathsf{G}, \mathsf{U}, \mathsf{W}, \mathsf{R}$

  according to following BNF grammar:

$$\varphi \ ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathsf{X}\,\varphi \mid \mathsf{F}\,\varphi \mid \mathsf{G}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \varphi\,\mathsf{W}\,\varphi \mid \varphi\,\mathsf{R}\,\varphi$$

Definitions

- LTL (linear-time temporal logic) formulas are built from
  - atoms $p, q, r, p_1, p_2, \ldots$
  - logical connectives $\bot, \top, \neg, \wedge, \vee, \rightarrow$
  - temporal connectives $\mathsf{X}, \mathsf{F}, \mathsf{G}, \mathsf{U}, \mathsf{W}, \mathsf{R}$

  according to following BNF grammar:

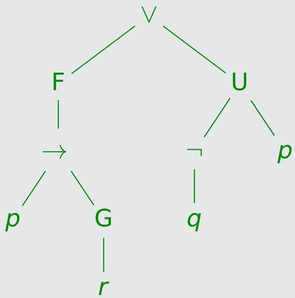
$$\varphi \ ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathsf{X}\,\varphi \mid \mathsf{F}\,\varphi \mid \mathsf{G}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \varphi\,\mathsf{W}\,\varphi \mid \varphi\,\mathsf{R}\,\varphi$$

- notational conventions:
  - binding precedence $\neg, \mathsf{X}, \mathsf{F}, \mathsf{G} \ > \ \mathsf{U}, \mathsf{W}, \mathsf{R} \ > \ \wedge, \vee \ > \ \rightarrow$
  - omit outer parentheses
  - $\rightarrow, \wedge, \vee$ are right-associative

Example

formula $\mathsf{F}\,(p \to \mathsf{G}\,r) \vee \neg q\,\mathsf{U}\,p$

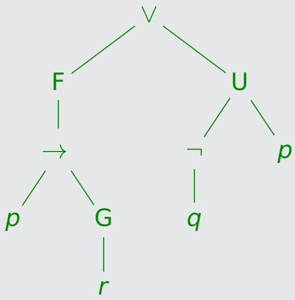
parse tree

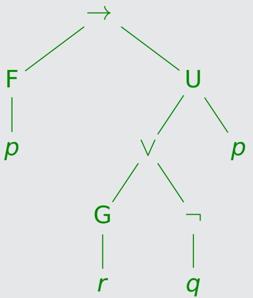


Example

formula $\mathsf{F}\,(p \rightarrow \mathsf{G}\,r) \vee \neg q\,\mathsf{U}\,p$ $\qquad$ $\mathsf{F}\,p \rightarrow (\mathsf{G}\,r \vee \neg q)\,\mathsf{U}\,p$

parse tree

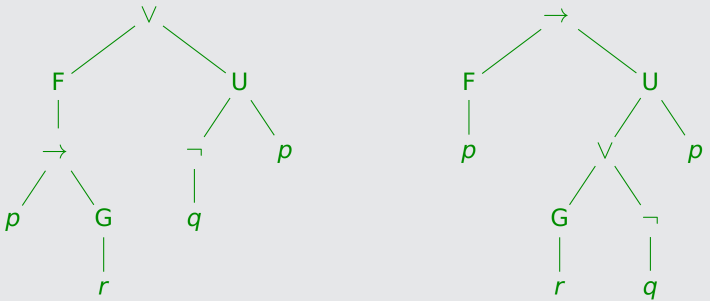




Example

formula $\mathsf{F}\,(p \rightarrow \mathsf{G}\,r) \vee \neg q\,\mathsf{U}\,p$ $\qquad$ $\mathsf{F}\,p \rightarrow (\mathsf{G}\,r \vee \neg q)\,\mathsf{U}\,p$

parse tree



X	next state	F	∃ future state	W	weak until
U	until	G	∀ states globally	R	release

Outline

Definition

transition system (model) is triple $\mathcal{M} = (S, \to, L)$ with

① set of states S

Definition

transition system (model) is triple $\mathcal{M} = (S, \rightarrow, L)$ with

① set of states S

② transition relation $\rightarrow \subseteq S \times S$ such that $\forall\, s \in S \;\; \exists\, t \in S$ with $s \rightarrow t$

Definition

transition system (model) is triple $\mathcal{M} = (S, \to, L)$ with

① set of states S

② transition relation $\to \subseteq S \times S$ such that $\forall s \in S \;\; \exists t \in S$ with $s \to t$ ("no deadlock")

Definition

transition system (model) is triple $\mathcal{M} = (S, \to, L)$ with

① set of states S

② transition relation $\to \subseteq S \times S$ such that $\forall\, s \in S \;\; \exists\, t \in S$ with $s \to t$ ("no deadlock")

③ labeling function $L\colon S \to 2^{\text{atoms}}$

Definition

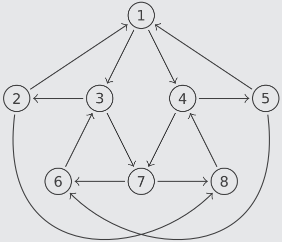
transition system (model) is triple $\mathcal{M} = (S, \to, L)$ with

① set of states S

② transition relation $\to \subseteq S \times S$ such that $\forall s \in S \;\; \exists t \in S$ with $s \to t$ ("no deadlock")

③ labeling function $L\colon S \to 2^{\text{atoms}}$

Example



model $\mathcal{M} = (S, \to, L)$

$$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$L(1) = \{I_A, I_B\}$ $\quad$ $L(5) = \{I_A, P_B\}$

$L(2) = \{P_A, I_B\}$ $\quad$ $L(6) = \{R_A, P_B\}$

$L(3) = \{R_A, I_B\}$ $\quad$ $L(7) = \{R_A, R_B\}$

$L(4) = \{I_A, R_B\}$ $\quad$ $L(8) = \{P_A, R_B\}$

Definition

- path in model $\mathcal{M} = (S, \rightarrow, L)$ is infinite sequence $s_1 \rightarrow s_2 \rightarrow \cdots$

Definition

- path in model $\mathcal{M} = (S, \rightarrow, L)$ is infinite sequence $s_1 \rightarrow s_2 \rightarrow \cdots$
- $\forall$ paths $\pi = s_1 \rightarrow s_2 \rightarrow \cdots$ $\forall\, i \geqslant 1$ $\pi^{i} = s_i \rightarrow s_{i+1} \rightarrow \cdots$

Definition

- path in model $\mathcal{M} = (S, \rightarrow, L)$ is infinite sequence $s_1 \rightarrow s_2 \rightarrow \cdots$
- $\forall$ paths $\pi = s_1 \rightarrow s_2 \rightarrow \cdots \quad \forall\, i \geqslant 1 \quad \pi^i = s_i \rightarrow s_{i+1} \rightarrow \cdots$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \rightarrow s_2 \rightarrow \cdots$ in model $\mathcal{M} = (S, \rightarrow, L)$

$$\pi \vDash \varphi$$

Definition

- path in model $\mathcal{M} = (S, \to, L)$ is infinite sequence $s_1 \to s_2 \to \cdots$
- $\forall$ paths $\pi = s_1 \to s_2 \to \cdots$ $\forall\, i \geqslant 1$ $\pi^i = s_i \to s_{i+1} \to \cdots$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\pi \vDash \top \qquad\qquad \pi \nvDash \bot$$

Definition

- path in model $\mathcal{M} = (S, \to, L)$ is infinite sequence $s_1 \to s_2 \to \cdots$
- $\forall$ paths $\pi = s_1 \to s_2 \to \cdots \quad \forall\, i \geqslant 1 \quad \pi^i = s_i \to s_{i+1} \to \cdots$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\pi \vDash \top \qquad\qquad \pi \nvDash \bot$$

$$\pi \vDash p \qquad \Longleftrightarrow \qquad p \in L(s_1)$$

Definition

- path in model $\mathcal{M} = (S, \to, L)$ is infinite sequence $s_1 \to s_2 \to \cdots$
- $\forall$ paths $\pi = s_1 \to s_2 \to \cdots \quad \forall\, i \geqslant 1 \quad \pi^i = s_i \to s_{i+1} \to \cdots$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\pi \vDash \top \qquad\qquad \pi \nvDash \bot$$

$$\pi \vDash p \quad\iff\quad p \in L(s_1)$$

$$\pi \vDash \neg\varphi \quad\iff\quad \pi \nvDash \varphi$$

Definition

- path in model $\mathcal{M} = (S, \to, L)$ is infinite sequence $s_1 \to s_2 \to \cdots$
- $\forall$ paths $\pi = s_1 \to s_2 \to \cdots \quad \forall\, i \geqslant 1 \quad \pi^i = s_i \to s_{i+1} \to \cdots$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

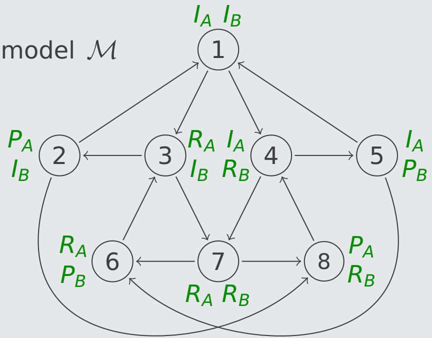
$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\begin{array}{llllll}
\pi \vDash \top & & \pi \nvDash \bot & \pi \vDash \varphi \wedge \psi & \iff & \pi \vDash \varphi \text{ and } \pi \vDash \psi \\
\pi \vDash p & \iff & p \in L(s_1) & \pi \vDash \varphi \vee \psi & \iff & \pi \vDash \varphi \text{ or } \pi \vDash \psi \\
\pi \vDash \neg\varphi & \iff & \pi \nvDash \varphi & \pi \vDash \varphi \to \psi & \iff & \pi \nvDash \varphi \text{ or } \pi \vDash \psi
\end{array}$$

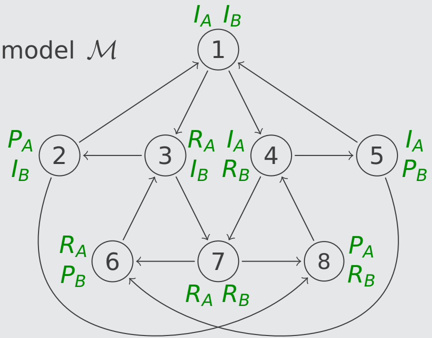
Example

model $\mathcal{M}$



Example

model $\mathcal{M}$

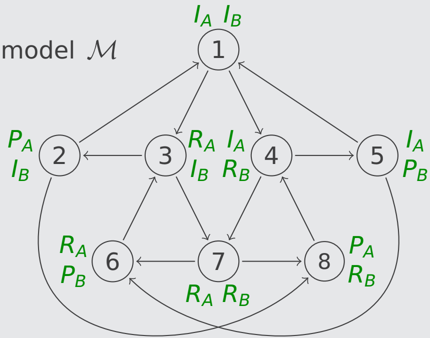


$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots$$

$$\pi_1 \models I_A$$

Example

model $\mathcal{M}$



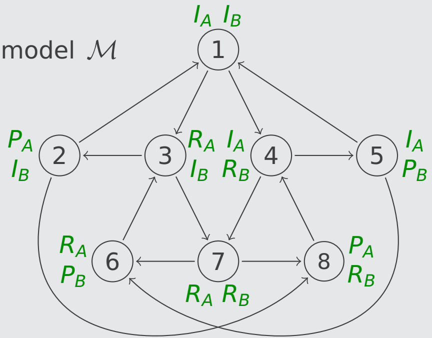
$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots$$

$\pi_1 \models I_A$

$\pi_1 \not\models R_A \wedge I_B$

Example

model $\mathcal{M}$



$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots$$

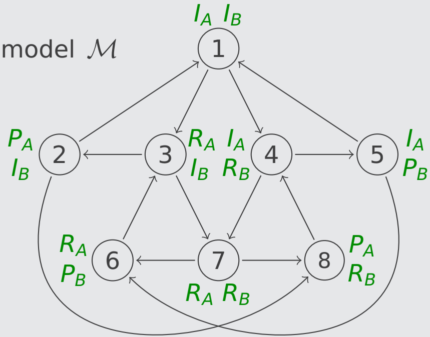
$\pi_1 \vDash I_A$

$\pi_1 \nvDash R_A \wedge I_B$

$\pi_1 \nvDash I_B \to P_A \vee R_B$

Example

model $\mathcal{M}$



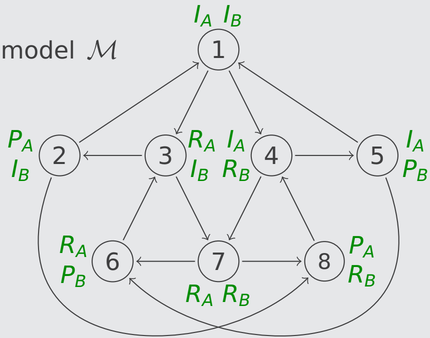
$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots$$

$$\pi_2 = 7 \to 6 \to 3 \to 7 \to 6 \to 3 \to \cdots$$

$\pi_1 \vDash I_A$	$\pi_1 \nvDash R_A \wedge I_B$	$\pi_1 \nvDash I_B \to P_A \vee R_B$
$\pi_2 \quad I_A$	$\pi_2 \quad R_A \wedge I_B$	$\pi_2 \quad I_B \to P_A \vee R_B$

Example

model $\mathcal{M}$



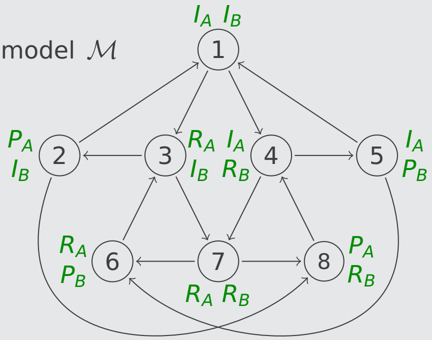
$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots$$

$$\pi_2 = 7 \to 6 \to 3 \to 7 \to 6 \to 3 \to \cdots$$

$\pi_1 \vDash I_A$	$\pi_1 \nvDash R_A \wedge I_B$	$\pi_1 \nvDash I_B \to P_A \vee R_B$
$\pi_2 \nvDash I_A$	$\pi_2 \quad R_A \wedge I_B$	$\pi_2 \quad I_B \to P_A \vee R_B$

Example

model $\mathcal{M}$



$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots$$

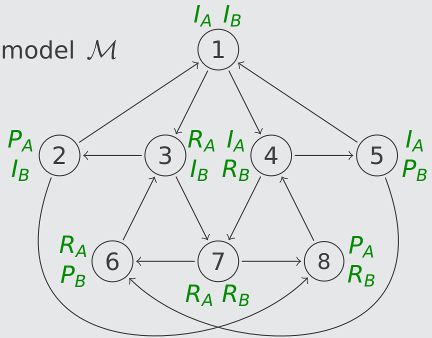
$$\pi_2 = 7 \to 6 \to 3 \to 7 \to 6 \to 3 \to \cdots$$

$\pi_1 \models I_A$ $\qquad$ $\pi_1 \not\models R_A \wedge I_B$ $\qquad$ $\pi_1 \not\models I_B \to P_A \vee R_B$

$\pi_2 \not\models I_A$ $\qquad$ $\pi_2^{6} \models R_A \wedge I_B$ $\qquad$ $\pi_2 \quad I_B \to P_A \vee R_B$

Example

model $\mathcal{M}$



$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots$$

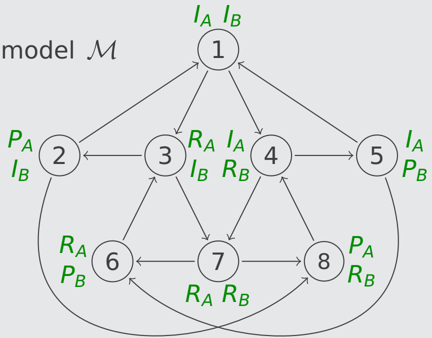
$$\pi_2 = 7 \to 6 \to 3 \to 7 \to 6 \to 3 \to \cdots$$

$\pi_1 \vDash I_A$ $\qquad$ $\pi_1 \nvDash R_A \wedge I_B$ $\qquad$ $\pi_1 \nvDash I_B \to P_A \vee R_B$

$\pi_2 \nvDash I_A$ $\qquad$ $\pi_2^6 \vDash R_A \wedge I_B$ $\qquad$ $\pi_2 \vDash I_B \to P_A \vee R_B$

Example

model $\mathcal{M}$



$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots = (1\,3\,2)^\omega$$

$$\pi_2 = 7 \to 6 \to 3 \to 7 \to 6 \to 3 \to \cdots = (7\,6\,3)^\omega$$

$\pi_1 \models I_A$ $\qquad$ $\pi_1 \not\models R_A \wedge I_B$ $\qquad$ $\pi_1 \not\models I_B \to P_A \vee R_B$

$\pi_2 \not\models I_A$ $\qquad$ $\pi_2^6 \models R_A \wedge I_B$ $\qquad$ $\pi_2 \models I_B \to P_A \vee R_B$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\pi \vDash \mathsf{X}\,\varphi \quad \iff \quad \pi^2 \vDash \varphi$$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\pi \vDash \mathsf{X}\,\varphi \quad\iff\quad \pi^2 \vDash \varphi$$

$$\pi \vDash \mathsf{F}\,\varphi \quad\iff\quad \exists\, i \geqslant 1 \quad \pi^i \vDash \varphi$$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\pi \vDash \mathsf{X}\,\varphi \iff \pi^2 \vDash \varphi$$

$$\pi \vDash \mathsf{F}\,\varphi \iff \exists\, i \geqslant 1 \quad \pi^i \vDash \varphi$$

$$\pi \vDash \mathsf{G}\,\varphi \iff \forall\, i \geqslant 1 \quad \pi^i \vDash \varphi$$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\begin{aligned}
\pi \vDash \mathsf{X}\,\varphi &\iff \pi^2 \vDash \varphi \\
\pi \vDash \mathsf{F}\,\varphi &\iff \exists\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi \vDash \mathsf{G}\,\varphi &\iff \forall\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi \vDash \varphi\,\mathsf{U}\,\psi &\iff \exists\, i \geqslant 1 \quad \pi^i \vDash \psi \;\text{ and }\; \forall\, j < i \quad \pi^j \vDash \varphi
\end{aligned}$$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \models \varphi$$

is defined by induction on φ:

$$\begin{aligned}
\pi \models \mathsf{X}\,\varphi &\iff \pi^2 \models \varphi \\
\pi \models \mathsf{F}\,\varphi &\iff \exists\, i \geqslant 1 \quad \pi^i \models \varphi \\
\pi \models \mathsf{G}\,\varphi &\iff \forall\, i \geqslant 1 \quad \pi^i \models \varphi \\
\pi \models \varphi\,\mathsf{U}\,\psi &\iff \exists\, i \geqslant 1 \quad \pi^i \models \psi \quad \text{and} \quad \forall\, j < i \quad \pi^j \models \varphi \\
\pi \models \varphi\,\mathsf{W}\,\psi &\iff \big(\exists\, i \geqslant 1 \quad \pi^i \models \psi \quad \text{and} \quad \forall\, j < i \quad \pi^j \models \varphi\big) \quad \text{or} \quad \forall\, i \geqslant 1 \quad \pi^i \models \varphi
\end{aligned}$$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

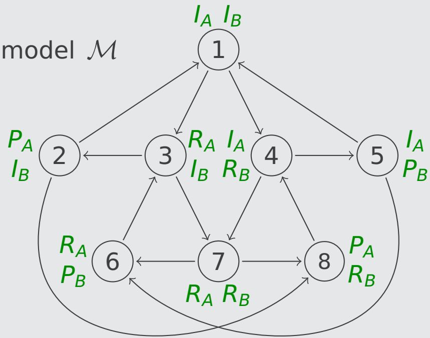
$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\begin{aligned}
\pi \vDash \mathsf{X}\,\varphi \quad &\Longleftrightarrow \quad \pi^2 \vDash \varphi \\
\pi \vDash \mathsf{F}\,\varphi \quad &\Longleftrightarrow \quad \exists\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi \vDash \mathsf{G}\,\varphi \quad &\Longleftrightarrow \quad \forall\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi \vDash \varphi\,\mathsf{U}\,\psi \quad &\Longleftrightarrow \quad \exists\, i \geqslant 1 \quad \pi^i \vDash \psi \quad \text{and} \quad \forall\, j < i \quad \pi^j \vDash \varphi \\
\pi \vDash \varphi\,\mathsf{W}\,\psi \quad &\Longleftrightarrow \quad \big(\exists\, i \geqslant 1 \quad \pi^i \vDash \psi \quad \text{and} \quad \forall\, j < i \quad \pi^j \vDash \varphi\big) \quad \text{or} \quad \forall\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi \vDash \varphi\,\mathsf{R}\,\psi \quad &\Longleftrightarrow \quad \big(\exists\, i \geqslant 1 \quad \pi^i \vDash \varphi \quad \text{and} \quad \forall\, j \leqslant i \quad \pi^j \vDash \psi\big) \quad \text{or} \quad \forall\, i \geqslant 1 \quad \pi^i \vDash \psi
\end{aligned}$$

Example

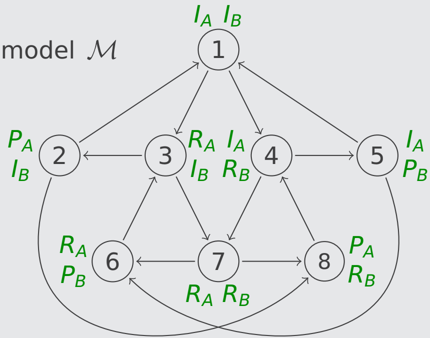
model $\mathcal{M}$



$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$

model $\mathcal{M}$

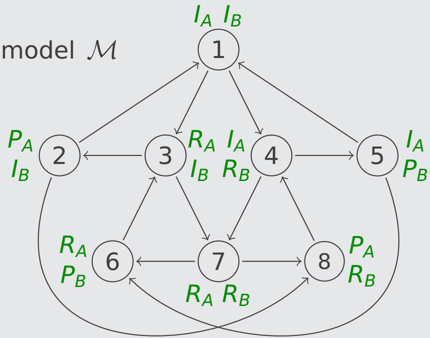


$\pi_1 = (1\,3\,2)^\omega$ $\qquad$ $\pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$ $\qquad$ $\pi_1 \quad \mathsf{F}\,P_A$ $\qquad$ $\pi_1 \quad \mathsf{X}\,\mathsf{X}\,P_B$

Example

model $\mathcal{M}$

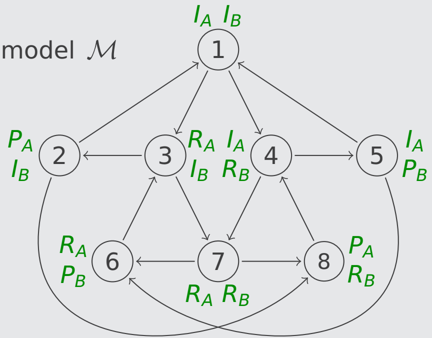


$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B) \qquad \pi_1 \vDash \mathsf{F}\, P_A \qquad \pi_1 \quad \mathsf{X}\,\mathsf{X}\, P_B$

Example

model $\mathcal{M}$

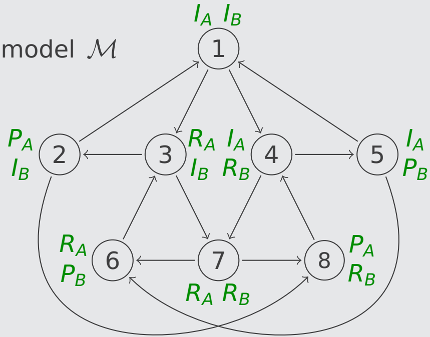


$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B) \qquad \pi_1 \vDash \mathsf{F}\, P_A \qquad \pi_1 \nvDash \mathsf{X}\,\mathsf{X}\, P_B$

Example

model $\mathcal{M}$



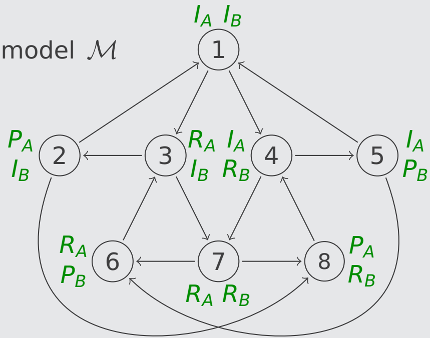
$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$ $\qquad$ $\pi_1 \vDash \mathsf{F}\,P_A$ $\qquad$ $\pi_1 \nvDash \mathsf{X}\,\mathsf{X}\,P_B$

$\pi_2 \qquad \mathsf{F}\,P_A$ $\qquad$ $\pi_2 \qquad \mathsf{G}\,\neg I_A$ $\qquad$ $\pi_2 \qquad \mathsf{G}\,\mathsf{F}\,P_B$

Example

model $\mathcal{M}$



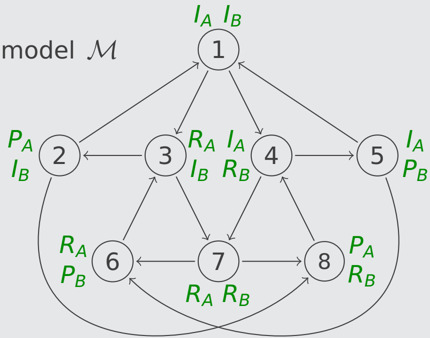
$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$ $\qquad$ $\pi_1 \vDash \mathsf{F}\,P_A$ $\qquad$ $\pi_1 \nvDash \mathsf{X}\,\mathsf{X}\,P_B$

$\pi_2 \nvDash \mathsf{F}\,P_A$ $\qquad$ $\pi_2 \quad \mathsf{G}\,\neg I_A$ $\qquad$ $\pi_2 \quad \mathsf{G}\,\mathsf{F}\,P_B$

Example

model $\mathcal{M}$



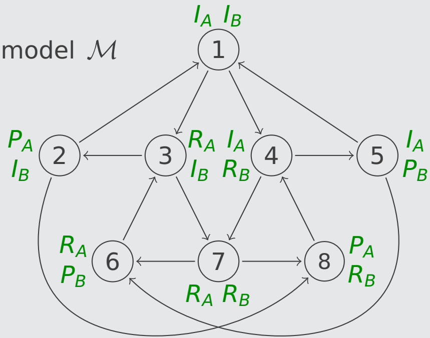
$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$ $\qquad$ $\pi_1 \vDash \mathsf{F}\,P_A$ $\qquad$ $\pi_1 \nvDash \mathsf{X}\,\mathsf{X}\,P_B$

$\pi_2 \nvDash \mathsf{F}\,P_A$ $\qquad$ $\pi_2 \vDash \mathsf{G}\,\neg I_A$ $\qquad$ $\pi_2 \quad \mathsf{G}\,\mathsf{F}\,P_B$

Example

model $\mathcal{M}$



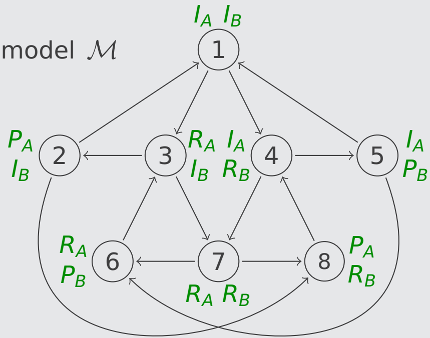
$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$ $\qquad$ $\pi_1 \vDash \mathsf{F}\,P_A$ $\qquad$ $\pi_1 \nvDash \mathsf{X}\,\mathsf{X}\,P_B$

$\pi_2 \nvDash \mathsf{F}\,P_A$ $\qquad$ $\pi_2 \vDash \mathsf{G}\,\neg I_A$ $\qquad$ $\pi_2 \vDash \mathsf{G}\,\mathsf{F}\,P_B$

Example

model $\mathcal{M}$



$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

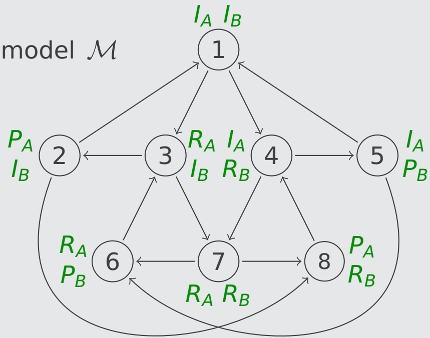
$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$ $\qquad \pi_1 \vDash \mathsf{F}\,P_A$ $\qquad \pi_1 \nvDash \mathsf{X}\,\mathsf{X}\,P_B$

$\pi_2 \nvDash \mathsf{F}\,P_A$ $\qquad \pi_2 \vDash \mathsf{G}\,\neg I_A$ $\qquad \pi_2 \vDash \mathsf{G}\,\mathsf{F}\,P_B$

$\pi_1 \quad I_A\,\mathsf{U}\,P_A$ $\qquad \pi_2 \quad \neg I_A\,\mathsf{W}\,P_A$ $\qquad \pi_2 \quad P_B\,\mathsf{R}\,R_B$

Example

model $\mathcal{M}$

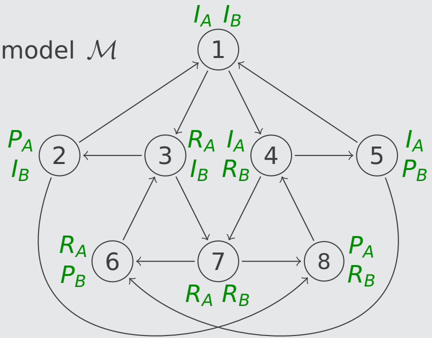


$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \models \mathsf{X}(R_A \vee R_B)$	$\pi_1 \models \mathsf{F}\, P_A$	$\pi_1 \not\models \mathsf{X}\,\mathsf{X}\, P_B$
$\pi_2 \not\models \mathsf{F}\, P_A$	$\pi_2 \models \mathsf{G}\, \neg I_A$	$\pi_2 \models \mathsf{G}\,\mathsf{F}\, P_B$
$\pi_1 \not\models I_A \,\mathsf{U}\, P_A$	$\pi_2 \quad \neg I_A \,\mathsf{W}\, P_A$	$\pi_2 \quad P_B \,\mathsf{R}\, R_B$

Example

model $\mathcal{M}$



$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \models \mathsf{X}(R_A \vee R_B)$

$\pi_1 \models \mathsf{F}\, P_A$

$\pi_1 \not\models \mathsf{X}\,\mathsf{X}\, P_B$

$\pi_2 \not\models \mathsf{F}\, P_A$

$\pi_2 \models \mathsf{G} \neg I_A$

$\pi_2 \models \mathsf{G}\,\mathsf{F}\, P_B$

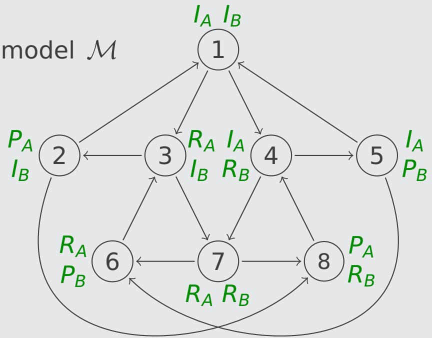
$\pi_1 \not\models I_A \,\mathsf{U}\, P_A$

$\pi_2 \models \neg I_A \,\mathsf{W}\, P_A$

$\pi_2 \qquad P_B \,\mathsf{R}\, R_B$

Example

model $\mathcal{M}$



$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \vDash \mathsf{X}(R_A \vee R_B)$ $\qquad$ $\pi_1 \vDash \mathsf{F}\,P_A$ $\qquad$ $\pi_1 \nvDash \mathsf{X}\,\mathsf{X}\,P_B$

$\pi_2 \nvDash \mathsf{F}\,P_A$ $\qquad$ $\pi_2 \vDash \mathsf{G}\,\neg I_A$ $\qquad$ $\pi_2 \vDash \mathsf{G}\,\mathsf{F}\,P_B$

$\pi_1 \nvDash I_A\,\mathsf{U}\,P_A$ $\qquad$ $\pi_2 \vDash \neg I_A\,\mathsf{W}\,P_A$ $\qquad$ $\pi_2 \nvDash P_B\,\mathsf{R}\,R_B$

Definition

model $\mathcal{M} = (S, \to, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \iff \forall$ paths $\pi = s \to \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

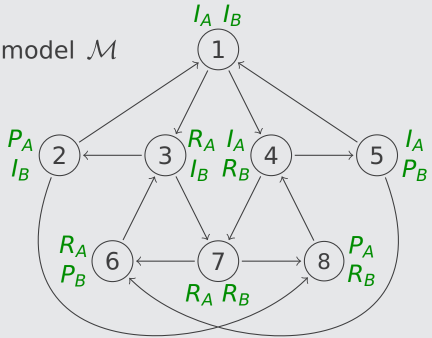
Definition

model $\mathcal{M} = (S, \rightarrow, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \iff \forall$ paths $\pi = s \rightarrow \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example

model $\mathcal{M}$



$\mathcal{M}, 1 \nvDash \mathrm{G}(R_A \rightarrow \mathrm{F}\, P_A)$

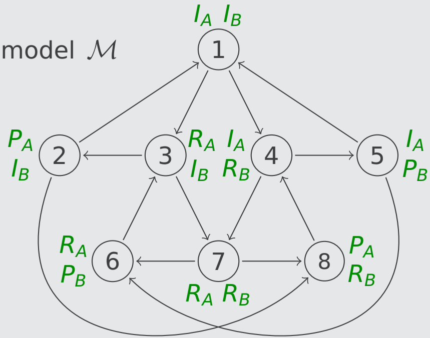
Definition

model $\mathcal{M} = (S, \rightarrow, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \quad \Longleftrightarrow \quad \forall$ paths $\pi = s \rightarrow \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example

model $\mathcal{M}$



$\mathcal{M}, 1 \nvDash \mathsf{G}(R_A \rightarrow \mathsf{F}\, P_A)$

$\mathcal{M}, 4 \qquad \neg(R_B \,\mathsf{U}\, P_B)$

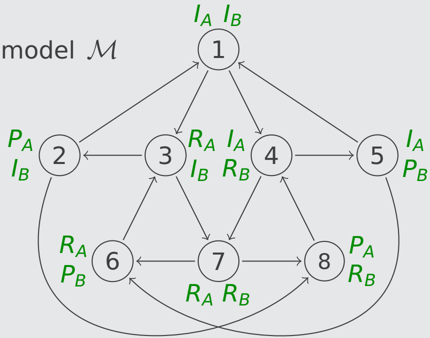
Definition

model $\mathcal{M} = (S, \rightarrow, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \quad \Longleftrightarrow \quad \forall$ paths $\pi = s \rightarrow \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example

model $\mathcal{M}$



$\mathcal{M}, 1 \nvDash \mathrm{G}(R_A \rightarrow \mathrm{F}\, P_A)$

$\mathcal{M}, 4 \nvDash \neg(R_B \,\mathrm{U}\, P_B)$

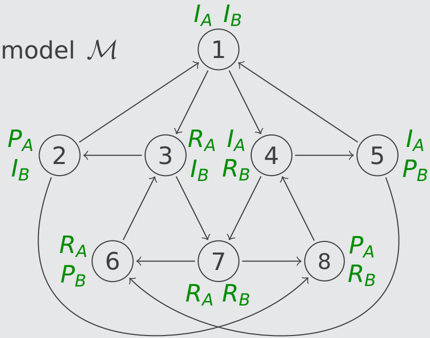
Definition

model $\mathcal{M} = (S, \to, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \iff \forall$ paths $\pi = s \to \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example

model $\mathcal{M}$



$\mathcal{M}, 1 \nvDash \mathsf{G}(R_A \to \mathsf{F}\, P_A)$

$\mathcal{M}, 4 \nvDash \neg(R_B \,\mathsf{U}\, P_B)$

$\mathcal{M}, 4 \qquad R_B \,\mathsf{U}\, P_B$

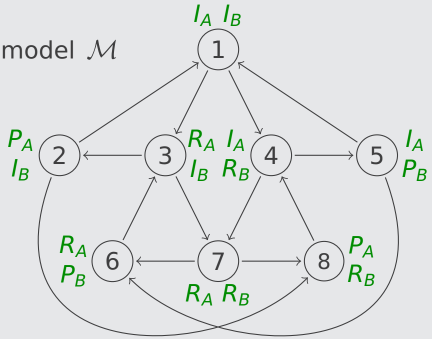
Definition

model $\mathcal{M} = (S, \rightarrow, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \iff \forall$ paths $\pi = s \rightarrow \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example

model $\mathcal{M}$



$\mathcal{M}, 1 \nvDash \mathsf{G}(R_A \rightarrow \mathsf{F}\, P_A)$

$\mathcal{M}, 4 \nvDash \neg(R_B \,\mathsf{U}\, P_B)$

$\mathcal{M}, 4 \nvDash R_B \,\mathsf{U}\, P_B$

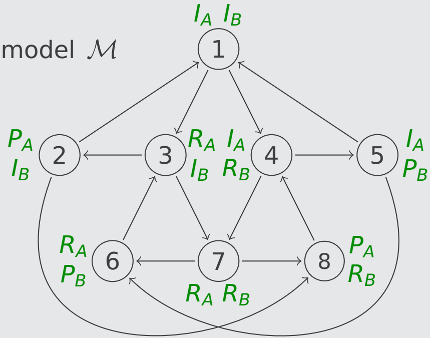
Definition

model $\mathcal{M} = (S, \to, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \iff \forall$ paths $\pi = s \to \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example

model $\mathcal{M}$



$\mathcal{M}, 1 \nvDash \mathrm{G}(R_A \to \mathrm{F}\, P_A)$

$\mathcal{M}, 4 \nvDash \neg(R_B \,\mathrm{U}\, P_B)$

$\mathcal{M}, 4 \nvDash R_B \,\mathrm{U}\, P_B$

$\mathcal{M}, 6 \quad \mathrm{X}(\mathrm{F}\, I_B \wedge ((\mathrm{X}\, \neg P_B)\, \mathrm{R}\, R_A))$

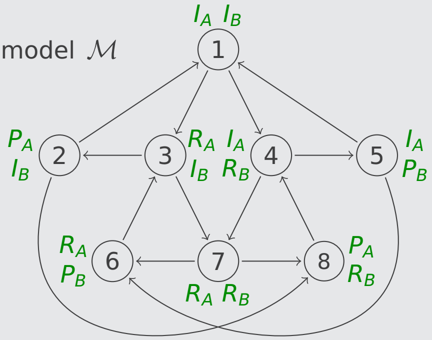
Definition

model $\mathcal{M} = (S, \to, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \iff \forall$ paths $\pi = s \to \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example

model $\mathcal{M}$



$\mathcal{M}, 1 \nvDash \mathsf{G}(R_A \to \mathsf{F}\, P_A)$

$\mathcal{M}, 4 \nvDash \neg(R_B \,\mathsf{U}\, P_B)$

$\mathcal{M}, 4 \nvDash \quad R_B \,\mathsf{U}\, P_B$

$\mathcal{M}, 6 \vDash \mathsf{X}(\mathsf{F}\, I_B \wedge ((\mathsf{X}\, \neg P_B)\, \mathsf{R}\, R_A))$

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$\forall$ models $\mathcal{M} = (S, \to, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$\forall$ models $\mathcal{M} = (S, \to, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Theorem

$$\neg \mathsf{X} \varphi \equiv \mathsf{X} \neg \varphi$$

Definition

LTL formulas φ and ψ are semantically equivalent $(\varphi \equiv \psi)$ if

$\forall$ models $\mathcal{M} = (S, \to, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \quad \Longleftrightarrow \quad \pi \vDash \psi$$

Theorem

$$\neg \mathsf{X}\,\varphi \equiv \mathsf{X} \neg\varphi$$

$$\neg \mathsf{F}\,\varphi \equiv \mathsf{G} \neg\varphi$$

$$\neg \mathsf{G}\,\varphi \equiv \mathsf{F} \neg\varphi$$

Definition

LTL formulas φ and ψ are semantically equivalent $(\varphi \equiv \psi)$ if

$\forall$ models $\mathcal{M} = (S, \rightarrow, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Theorem

$$\neg\, \mathsf{X}\, \varphi \equiv \mathsf{X} \neg \varphi$$

$$\neg\, \mathsf{F}\, \varphi \equiv \mathsf{G} \neg \varphi$$

$$\neg\, \mathsf{G}\, \varphi \equiv \mathsf{F} \neg \varphi$$

$$\neg (\varphi \,\mathsf{U}\, \psi) \equiv \neg \varphi \,\mathsf{R}\, \neg \psi$$

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$\forall$ models $\mathcal{M} = (S, \rightarrow, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Theorem

$$\neg\, \mathsf{X}\, \varphi \equiv \mathsf{X} \neg \varphi$$

$$\neg\, \mathsf{F}\, \varphi \equiv \mathsf{G} \neg \varphi$$

$$\neg\, \mathsf{G}\, \varphi \equiv \mathsf{F} \neg \varphi$$

$$\neg(\varphi\, \mathsf{U}\, \psi) \equiv \neg \varphi\, \mathsf{R} \neg \psi$$

$$\neg(\varphi\, \mathsf{R}\, \psi) \equiv \neg \varphi\, \mathsf{U} \neg \psi$$

$$\mathsf{F}(\varphi \vee \psi) \equiv \mathsf{F}\, \varphi \vee \mathsf{F}\, \psi$$

$$\mathsf{G}(\varphi \wedge \psi) \equiv \mathsf{G}\, \varphi \wedge \mathsf{G}\, \psi$$

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$\forall$ models $\mathcal{M} = (S, \rightarrow, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Theorem

$$\neg \mathsf{X}\, \varphi \equiv \mathsf{X} \neg \varphi$$

$$\neg \mathsf{F}\, \varphi \equiv \mathsf{G} \neg \varphi \qquad \mathsf{F}(\varphi \vee \psi) \equiv \mathsf{F}\, \varphi \vee \mathsf{F}\, \psi$$

$$\neg \mathsf{G}\, \varphi \equiv \mathsf{F} \neg \varphi \qquad \mathsf{G}(\varphi \wedge \psi) \equiv \mathsf{G}\, \varphi \wedge \mathsf{G}\, \psi$$

$$\neg(\varphi\, \mathsf{U}\, \psi) \equiv \neg \varphi\, \mathsf{R} \neg \psi \qquad \mathsf{F}\, \varphi \equiv \top\, \mathsf{U}\, \varphi$$

$$\neg(\varphi\, \mathsf{R}\, \psi) \equiv \neg \varphi\, \mathsf{U} \neg \psi \qquad \mathsf{G}\, \varphi \equiv \bot\, \mathsf{R}\, \varphi$$

Definition

LTL formulas φ and ψ are semantically equivalent $(\varphi \equiv \psi)$ if

$\forall$ models $\mathcal{M} = (S, \to, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Theorem

$$\neg \mathsf{X}\, \varphi \equiv \mathsf{X} \neg \varphi$$

$$\neg \mathsf{F}\, \varphi \equiv \mathsf{G} \neg \varphi$$

$$\neg \mathsf{G}\, \varphi \equiv \mathsf{F} \neg \varphi$$

$$\neg(\varphi \,\mathsf{U}\, \psi) \equiv \neg\varphi \,\mathsf{R}\, \neg\psi$$

$$\neg(\varphi \,\mathsf{R}\, \psi) \equiv \neg\varphi \,\mathsf{U}\, \neg\psi$$

$$\varphi \,\mathsf{U}\, \psi \equiv \varphi \,\mathsf{W}\, \psi \wedge \mathsf{F}\, \psi$$

$$\varphi \,\mathsf{W}\, \psi \equiv \varphi \,\mathsf{U}\, \psi \vee \mathsf{G}\, \varphi$$

$$\mathsf{F}(\varphi \vee \psi) \equiv \mathsf{F}\, \varphi \vee \mathsf{F}\, \psi$$

$$\mathsf{G}(\varphi \wedge \psi) \equiv \mathsf{G}\, \varphi \wedge \mathsf{G}\, \psi$$

$$\mathsf{F}\, \varphi \equiv \top \,\mathsf{U}\, \varphi$$

$$\mathsf{G}\, \varphi \equiv \bot \,\mathsf{R}\, \varphi$$

Definition

LTL formulas φ and ψ are semantically equivalent $(\varphi \equiv \psi)$ if

$$\forall \text{ models } \mathcal{M} = (S, \to, L) \qquad \pi \vDash \varphi \iff \pi \vDash \psi$$
$$\forall \text{ paths } \pi \text{ in } \mathcal{M}$$

Theorem

$$\neg \mathsf{X}\,\varphi \equiv \mathsf{X}\,\neg\varphi$$

$$\neg \mathsf{F}\,\varphi \equiv \mathsf{G}\,\neg\varphi \qquad\qquad \mathsf{F}(\varphi \vee \psi) \equiv \mathsf{F}\,\varphi \vee \mathsf{F}\,\psi$$

$$\neg \mathsf{G}\,\varphi \equiv \mathsf{F}\,\neg\varphi \qquad\qquad \mathsf{G}(\varphi \wedge \psi) \equiv \mathsf{G}\,\varphi \wedge \mathsf{G}\,\psi$$

$$\neg(\varphi\,\mathsf{U}\,\psi) \equiv \neg\varphi\,\mathsf{R}\,\neg\psi \qquad\qquad \mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$$

$$\neg(\varphi\,\mathsf{R}\,\psi) \equiv \neg\varphi\,\mathsf{U}\,\neg\psi \qquad\qquad \mathsf{G}\,\varphi \equiv \bot\,\mathsf{R}\,\varphi$$

$$\varphi\,\mathsf{U}\,\psi \equiv \varphi\,\mathsf{W}\,\psi \wedge \mathsf{F}\,\psi \qquad\qquad \varphi\,\mathsf{W}\,\psi \equiv \psi\,\mathsf{R}\,(\varphi \vee \psi)$$

$$\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi \qquad\qquad \varphi\,\mathsf{R}\,\psi \equiv \psi\,\mathsf{W}\,(\varphi \wedge \psi)$$

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$\forall$ models $\mathcal{M} = (S, \to, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Theorem

$$\neg \mathsf{X}\, \varphi \equiv \mathsf{X} \neg \varphi$$

$$\neg \mathsf{F}\, \varphi \equiv \mathsf{G} \neg \varphi$$

$$\neg \mathsf{G}\, \varphi \equiv \mathsf{F} \neg \varphi$$

$$\neg(\varphi\, \mathsf{U}\, \psi) \equiv \neg \varphi\, \mathsf{R} \neg \psi$$

$$\neg(\varphi\, \mathsf{R}\, \psi) \equiv \neg \varphi\, \mathsf{U} \neg \psi$$

$$\varphi\, \mathsf{U}\, \psi \equiv \varphi\, \mathsf{W}\, \psi \wedge \mathsf{F}\, \psi$$

$$\varphi\, \mathsf{W}\, \psi \equiv \varphi\, \mathsf{U}\, \psi \vee \mathsf{G}\, \varphi$$

$$\varphi\, \mathsf{U}\, \psi \equiv \neg(\neg \psi\, \mathsf{U}\, (\neg \varphi \wedge \neg \psi)) \wedge \mathsf{F}\, \psi$$

$$\mathsf{F}(\varphi \vee \psi) \equiv \mathsf{F}\, \varphi \vee \mathsf{F}\, \psi$$

$$\mathsf{G}(\varphi \wedge \psi) \equiv \mathsf{G}\, \varphi \wedge \mathsf{G}\, \psi$$

$$\mathsf{F}\, \varphi \equiv \top\, \mathsf{U}\, \varphi$$

$$\mathsf{G}\, \varphi \equiv \bot\, \mathsf{R}\, \varphi$$

$$\varphi\, \mathsf{W}\, \psi \equiv \psi\, \mathsf{R}\, (\varphi \vee \psi)$$

$$\varphi\, \mathsf{R}\, \psi \equiv \psi\, \mathsf{W}\, (\varphi \wedge \psi)$$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi))$

Theorem

$$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad\Longleftrightarrow\quad \pi \nvDash \neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)$$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$\begin{aligned}
\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad &\iff \quad \pi \nvDash \neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi) \\
&\iff \quad \text{not} \;\; \exists\, i \geqslant 1 \;\; \left(\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi\right)
\end{aligned}$$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$
\begin{aligned}
\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad &\Longleftrightarrow \quad \pi \nvDash \neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi) \\
&\Longleftrightarrow \quad \text{not} \;\; \exists\, i \geqslant 1 \;\; \left(\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi\right) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; \text{not} \;\; \left(\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi\right)
\end{aligned}
$$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$
\begin{aligned}
\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad &\Longleftrightarrow \quad \pi \nvDash \neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi) \\
&\Longleftrightarrow \quad \text{not} \;\; \exists\, i \geqslant 1 \;\; \left(\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi\right) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; \text{not} \;\; \left(\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi\right) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; \left(\pi^i \nvDash \neg\varphi \wedge \neg\psi \;\; \text{or} \;\; \text{not} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi\right)
\end{aligned}
$$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$
\begin{aligned}
\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad &\Longleftrightarrow \quad \pi \nvDash \neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi) \\
&\Longleftrightarrow \quad \text{not} \;\; \exists\, i \geqslant 1 \;\; (\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; \text{not} \;\; (\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; (\pi^i \nvDash \neg\varphi \wedge \neg\psi \;\; \text{or} \;\; \text{not} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; (\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \nvDash \neg\psi)
\end{aligned}
$$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$
\begin{aligned}
\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad &\Longleftrightarrow \quad \pi \nvDash \neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi) \\
&\Longleftrightarrow \quad \text{not} \;\; \exists\, i \geqslant 1 \;\; (\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; \text{not} \;\; (\pi^i \vDash \neg\varphi \wedge \neg\psi \;\; \text{and} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; (\pi^i \nvDash \neg\varphi \wedge \neg\psi \;\; \text{or} \;\; \text{not} \;\; \forall\, j < i \;\; \pi^j \vDash \neg\psi) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; (\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \nvDash \neg\psi) \\
&\Longleftrightarrow \quad \forall\, i \geqslant 1 \;\; (\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \vDash \psi)
\end{aligned}
$$

Theorem

$$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \;\; \left(\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \vDash \psi\right)$$

Theorem

$\varphi \, \mathsf{U} \, \psi \equiv \neg(\neg\psi \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F} \, \psi$

$$\pi \vDash \neg(\neg\psi \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \; \left(\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\, \psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall i \geqslant 1 \;\; \left(\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists j < i \;\; \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \,\mathsf{U}\, \psi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\, \psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \ \left(\pi^i \vDash \varphi \vee \psi \ \text{ or } \ \exists\, j < i \ \ \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \,\mathsf{U}\, \psi$ and consider smallest m such that $\pi^m \vDash \psi$

Theorem

$\varphi \, \mathsf{U} \, \psi \equiv \neg(\neg\psi \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F} \, \psi$

$$\pi \vDash \neg(\neg\psi \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \;\; (\pi^i \vDash \varphi \vee \psi \;\text{ or }\; \exists\, j < i \;\; \pi^j \vDash \psi)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \, \mathsf{U} \, \psi$ and consider smallest m such that $\pi^m \vDash \psi$
- $\pi \vDash \mathsf{F} \, \psi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\, \psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \ \left(\pi^i \vDash \varphi \vee \psi \ \text{ or } \ \exists\, j < i \ \ \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \,\mathsf{U}\, \psi$ and consider smallest m such that $\pi^m \vDash \psi$
- $\pi \vDash \mathsf{F}\, \psi$ and $\pi^i \vDash \varphi$ for all $i < m$

Theorem

$\varphi \mathsf{U} \psi \equiv \neg(\neg\psi \mathsf{U} (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\psi$

$$\pi \vDash \neg(\neg\psi \mathsf{U} (\neg\varphi \wedge \neg\psi)) \iff \forall i \geqslant 1 \ \left(\pi^i \vDash \varphi \vee \psi \ \text{ or } \ \exists j < i \ \ \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \mathsf{U} \psi$ and consider smallest m such that $\pi^m \vDash \psi$
- $\pi \vDash \mathsf{F}\psi$ and $\pi^i \vDash \varphi$ for all $i < m$
- consider $i \geqslant 1$

Theorem

$$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad\Longleftrightarrow\quad \forall\, i \geqslant 1 \;\; \left(\pi^i \vDash \varphi \vee \psi \;\text{ or }\; \exists\, j < i \;\; \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \,\mathsf{U}\, \psi$ and consider smallest m such that $\pi^m \vDash \psi$
- $\pi \vDash \mathsf{F}\,\psi$ and $\pi^i \vDash \varphi$ for all $i < m$
- consider $i \geqslant 1$
  - if $i > m$ then $\exists\, j < i \;\; \pi^j \vDash \psi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\, \psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \ \left(\pi^i \vDash \varphi \vee \psi \ \text{ or } \ \exists\, j < i \ \ \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \,\mathsf{U}\, \psi$ and consider smallest m such that $\pi^m \vDash \psi$
- $\pi \vDash \mathsf{F}\, \psi$ and $\pi^i \vDash \varphi$ for all $i < m$
- consider $i \geqslant 1$
  - if $i > m$ then $\exists\, j < i \ \ \pi^j \vDash \psi$ and if $i \leqslant m$ then $\pi^i \vDash \varphi \vee \psi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \;\; (\pi^i \vDash \varphi \vee \psi \;\text{ or }\; \exists\, j < i \;\; \pi^j \vDash \psi)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \varphi \,\mathsf{U}\, \psi$ and consider smallest m such that $\pi^m \vDash \psi$
- $\pi \vDash \mathsf{F}\,\psi$ and $\pi^i \vDash \varphi$ for all $i < m$
- consider $i \geqslant 1$
  - if $i > m$ then $\exists\, j < i \;\; \pi^j \vDash \psi$ and if $i \leqslant m$ then $\pi^i \vDash \varphi \vee \psi$
- $\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi))$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \;\; \left(\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \quad\Longleftrightarrow\quad \forall\, i \geqslant 1 \;\; \left(\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$
- there exists smallest m such that $\pi^m \vDash \psi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\, \psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \;\; \left(\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\, \psi$
- there exists smallest m such that $\pi^m \vDash \psi$
- if $i < m$ then $\pi^i \vDash \neg\psi$

Theorem

$\varphi \, \mathsf{U} \, \psi \equiv \neg(\neg\psi \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F} \, \psi$

$$\pi \vDash \neg(\neg\psi \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \;\; \left(\pi^i \vDash \varphi \vee \psi \;\; \text{or} \;\; \exists\, j < i \;\; \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \neg(\neg\psi \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F} \, \psi$
- there exists smallest m such that $\pi^m \vDash \psi$
- if $i < m$ then $\pi^i \vDash \neg\psi$ and hence $\pi^i \vDash \varphi$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

$$\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \iff \forall\, i \geqslant 1 \ \left(\pi^i \vDash \varphi \vee \psi \ \text{ or } \ \exists\, j < i \ \ \pi^j \vDash \psi\right)$$

Proof

- consider arbitrary path $\pi = s_1 \to s_2 \to s_3 \to \cdots$ in arbitrary model $\mathcal{M}$
- suppose $\pi \vDash \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$
- there exists smallest m such that $\pi^m \vDash \psi$
- if $i < m$ then $\pi^i \vDash \neg\psi$ and hence $\pi^i \vDash \varphi$
- $\pi \vDash \varphi \,\mathsf{U}\, \psi$

Outline

Theorem

$\{\mathsf{X}, \mathsf{U}\}$, $\{\mathsf{X}, \mathsf{W}\}$ and $\{\mathsf{X}, \mathsf{R}\}$ are adequate sets of temporal connectives for LTL

Theorem

$\{X, U\}$, $\{X, W\}$ and $\{X, R\}$ are adequate sets of temporal connectives for LTL

Proof

- $F\,\varphi \equiv \top\,U\,\varphi$
- $G\,\varphi \equiv \neg\,F\,\neg\varphi$
- $\varphi\,R\,\psi \equiv \neg(\neg\varphi\,U\,\neg\psi)$
- $\varphi\,W\,\psi \equiv \varphi\,U\,\psi \vee G\,\varphi$

Theorem

$\{\mathsf{X}, \mathsf{U}\}$, $\{\mathsf{X}, \mathsf{W}\}$ and $\{\mathsf{X}, \mathsf{R}\}$ are adequate sets of temporal connectives for LTL

Proof

- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$
- $\varphi\,\mathsf{R}\,\psi \equiv \neg(\neg\varphi\,\mathsf{U}\,\neg\psi)$
- $\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi$
- $\varphi\,\mathsf{R}\,\psi \equiv \psi\,\mathsf{W}\,(\varphi \wedge \psi)$
- $\varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\varphi\,\mathsf{R}\,\neg\psi)$
- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$

Theorem

$\{\mathsf{X}, \mathsf{U}\}$, $\{\mathsf{X}, \mathsf{W}\}$ and $\{\mathsf{X}, \mathsf{R}\}$ are adequate sets of temporal connectives for LTL

Proof

- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$
- $\varphi\,\mathsf{R}\,\psi \equiv \neg(\neg\varphi\,\mathsf{U}\,\neg\psi)$
- $\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi$

- $\varphi\,\mathsf{R}\,\psi \equiv \psi\,\mathsf{W}\,(\varphi \wedge \psi)$
- $\varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\varphi\,\mathsf{R}\,\neg\psi)$
- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$

- $\varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\varphi\,\mathsf{R}\,\neg\psi)$
- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$
- $\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi$

Theorem

$\{\mathsf{X}, \mathsf{U}\}$, $\{\mathsf{X}, \mathsf{W}\}$ and $\{\mathsf{X}, \mathsf{R}\}$ are adequate sets of temporal connectives for LTL

Proof

- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$
- $\varphi\,\mathsf{R}\,\psi \equiv \neg(\neg\varphi\,\mathsf{U}\,\neg\psi)$
- $\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi$

- $\varphi\,\mathsf{R}\,\psi \equiv \psi\,\mathsf{W}\,(\varphi \wedge \psi)$
- $\varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\varphi\,\mathsf{R}\,\neg\psi)$
- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$

- $\varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\varphi\,\mathsf{R}\,\neg\psi)$
- $\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
- $\mathsf{G}\,\varphi \equiv \neg\,\mathsf{F}\,\neg\varphi$
- $\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi$

Outline

Esparza and Blondin

- Sections 11.1 and 13.2 of Automata Theory: An Algorithmic Approach (MIT Press 2023)

Esparza and Blondin

- Sections 11.1 and 13.2 of Automata Theory: An Algorithmic Approach (MIT Press 2023)

Important Concepts

- adequacy
- generalized Büchi automaton
- F
- G
- GBA
- linear–time temporal logic
- LTL
- R
- U
- W
- X

Esparza and Blondin

- Sections 11.1 and 13.2 of Automata Theory: An Algorithmic Approach (MIT Press 2023)

Important Concepts

- adequacy
- generalized Büchi automaton
- F
- G
- GBA
- linear–time temporal logic
- LTL
- R
- U
- W
- X

homework for January 9