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Automata and Logic

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Outline

Theorem

every DBA M can be effectively transformed into NBA M' such that $L(M') = \sim L(M)$

Theorem

ω–regular sets are closed under complement

Notation

for NBA $M = (Q, \Sigma, \Delta, S, F)$ and states $p, q \in Q$

$$L_{pq} = \{x \in \Sigma^* \mid q \in \widehat{\Delta}(\{p\}, x)\} \qquad L^{\mathsf{f}}_{pq} = \bigcup_{f \in F} L_{pf} \cdot L_{fq}$$

Definition

relation $\sim_M$ on Σ^* for NBA $M = (Q, \Sigma, \Delta, S, F)$: $u \sim_M v$ if for all $p, q \in Q$

$$u \in L_{pq} \iff v \in L_{pq} \quad \text{and} \quad u \in L^{\mathsf{f}}_{pq} \iff v \in L^{\mathsf{f}}_{pq}$$

Lemma

1. $\sim_M$ is equivalence relation of finite index
2. each equivalence class of $\sim_M$ is regular

Lemma

for all $x \in U \cdot V^\omega$ with equivalence classes U and V of $\sim_M$

$$x \in L(M) \implies U \cdot V^\omega \subseteq L(M) \qquad x \notin L(M) \implies U \cdot V^\omega \subseteq \sim L(M)$$

Lemma

for all $x \in \Sigma^\omega$ there exist equivalence classes U and V of $\sim_M$ such that $x \in U \cdot V^\omega$

Corollary

$$\sim L(M) = \bigcup \left\{ U \cdot V^\omega \;\middle|\; U \text{ and } V \text{ are equivalence classes of } \sim_M \text{ such that } U \cdot V^\omega \cap L(M) = \varnothing \right\}$$

Definitions

- first–order variables $V_1 = \{x, y, \ldots\}$ ranging over natural numbers
- second–order variables $V_2 = \{X, Y, \ldots\}$ ranging over sets of natural numbers
- formulas of monadic second–order logic (MSO)

$$\varphi ::= \bot \mid x < y \mid X(x) \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists x.\, \varphi \mid \exists X.\, \varphi$$

  with $x, y \in V_1$ and $X \in V_2$

- assignment α is mapping from variables $x \in V_1$ to $\mathbb{N}$ and $X \in V_2$ to subsets of $\mathbb{N}$
- assignment α satisfies formula φ ($\alpha \vDash \varphi$):

$$\alpha \nvDash \bot$$

$$\alpha \vDash x < y \iff \alpha(x) < \alpha(y) \qquad \alpha \vDash \neg\varphi \iff \alpha \nvDash \varphi$$

$$\alpha \vDash X(x) \iff \alpha(x) \in \alpha(X) \qquad \alpha \vDash \varphi_1 \vee \varphi_2 \iff \alpha \vDash \varphi_1 \text{ or } \alpha \vDash \varphi_2$$

$$\alpha \vDash \exists x.\, \varphi \iff \alpha[x \mapsto n] \vDash \varphi \text{ for some } n \in \mathbb{N}$$

$$\alpha \vDash \exists X.\, \varphi \iff \alpha[X \mapsto N] \vDash \varphi \text{ for some subset } N \subseteq \mathbb{N}$$

Definitions

- assignment α for MSO formula φ with $\mathrm{FV}(\varphi) = (x_1, \ldots, x_m, X_1, \ldots, X_n)$ is encoded as infinite string $\underline{\alpha} \in (\{0,1\}^{m+n})^\omega$:

$$\alpha(x_i) = j \qquad \text{if } i\text{–th entry of } j\text{–th symbol in } \underline{\alpha} \text{ is } 1$$

$$\alpha(X_i) = \{\, j \mid (m+i)\text{–th entry of } j\text{–th symbol in } \underline{\alpha} \text{ is } 1 \,\}$$

- infinite string over $\{0,1\}^{m+n}$ is *m–admissible* if first m rows contain exactly one 1 each
- $L_a(\varphi) = \{x \in (\{0,1\}^{m+n})^\omega \mid x \text{ is } m\text{–admissible and } \underline{x} \vDash \varphi\}$

Theorem

$L_a(\varphi)$ is ω–regular for every MSO formula φ

Theorem

set $A \subseteq \Sigma^\omega$ is ω–regular if and only if A is MSO definable

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

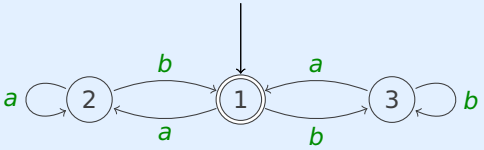
- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

Outline

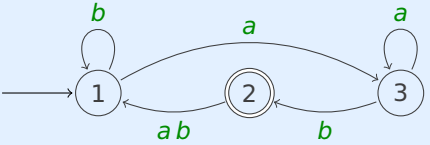
Remark

minimal DBAs are not unique

- $L = \{x \in \{a,b\}^\omega \mid |x|_a = \infty \text{ and } |x|_b = \infty\}$
- $L = L(M_1)$ for DBA M_1



- $L = L(M_2)$ for DBA M_2



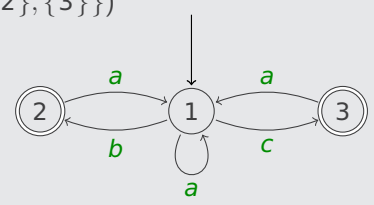
- M_1 and M_2 are not isomorphic
- no DBA with 2 states accepts L

Definition

- generalized Büchi automaton (GBA) is automaton $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$ with $F_1, \ldots, F_k \subseteq Q$
- $\infty(r) = \{q \in Q \mid q \text{ occurs infinitely often in run } r\}$
- run r of GBA is accepting if $\infty(r) \cap F_i \neq \varnothing$ for all $1 \leqslant i \leqslant k$

Example

GBA $M = (\{1,2,3\}, \{a,b,c\}, \Delta, \{1\}, \{\{2\},\{3\}\})$



$L(M) = \{x \in \{a,b,c\}^\omega \mid |x|_b = |x|_c = \infty \text{ and each } b \text{ and } c \text{ in } x \text{ is followed by } a\}$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

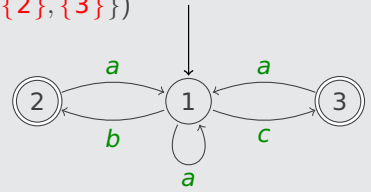
every GBA can be transformed into equivalent NBA

Proof

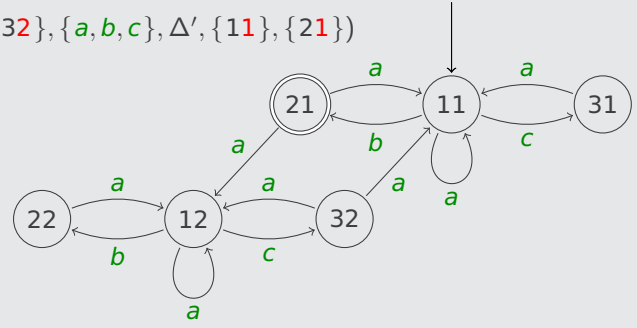
- GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \ldots, F_k\})$
- $L(M) = L(M')$ for NBA $M' = (Q', \Sigma, \Delta', S', F')$ with
  - $Q' = Q \times \{1, \ldots, k\}$
  - $S' = S \times \{1\}$
  - $\Delta'((q,i),a) = \Delta(q,a) \times \{j\}$ with $j = \begin{cases} i & \text{if } q \notin F_i \\ (i \bmod k) + 1 & \text{if } q \in F_i \end{cases}$
  - $F' = F_1 \times \{1\}$

Example

GBA $M = (\{1,2,3\}, \{a,b,c\}, \Delta, \{1\}, \{\{2\},\{3\}\})$



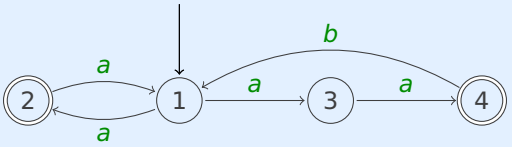
NBA $M' = (\{11,21,31,12,22,32\}, \{a,b,c\}, \Delta', \{11\}, \{21\})$



Outline

Question

Which statements about the following GBA $M = (\{1,2,3,4\}, \{a,b\}, \Delta, \{1\}, F)$ are true ?



A $L(M) = \{aa, aab\}^\omega$ if $F = \{\{2\}, \{3\}\}$

B $L(M) = \{aa, aab\}^\omega$ if $F = \{\{2,3\}\}$

C it does not matter whether state 3 or 4 is final

D there is only one possible choice for F such that $L(M) = \{aa, aab\}^\omega$

Outline

Formal Verification comprises

- framework for modeling systems (description language)
- specification language for describing properties to be verified
- verification method to establish whether description of system satisfies specification

Model Checking

automatic formal verification approach for concurrent systems based on temporal logic

Temporal Logic

- formulas are not statically true or false in model
- models of temporal logic contain several states and truth is dynamic
- formula can be true in some states and false in others

Model Checking

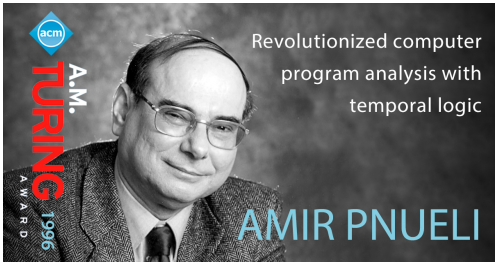
- models are transition systems $\mathcal{M}$
- properties are formulas φ in temporal logic
- model checker determines whether $\mathcal{M} \models \varphi$ is true or not

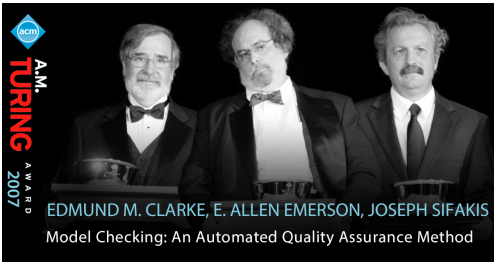
Two Temporal Logics

- computation tree logic (CTL)
- linear–time temporal logic (LTL)

Impact

both logics have been proven to be extremely fruitful in verifying hardware and communication protocols, and are increasingly applied to software verification





ACM Turing Awards

1996 Amir Pnueli

2007 Edmund M. Clarke, E. Allen Emerson, Joseph Sifakis

Outline

Definitions

- LTL (linear–time temporal logic) formulas are built from
  - atoms $p, q, r, p_1, p_2, \ldots$
  - logical connectives $\bot, \top, \neg, \wedge, \vee, \rightarrow$
  - temporal connectives $\mathsf{X}, \mathsf{F}, \mathsf{G}, \mathsf{U}, \mathsf{W}, \mathsf{R}$

  according to following BNF grammar:

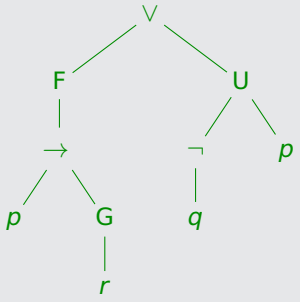
$$\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathsf{X}\varphi \mid \mathsf{F}\varphi \mid \mathsf{G}\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \varphi\,\mathsf{W}\,\varphi \mid \varphi\,\mathsf{R}\,\varphi$$

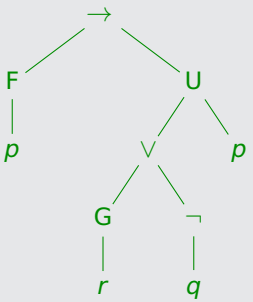
- notational conventions:
  - binding precedence $\neg, \mathsf{X}, \mathsf{F}, \mathsf{G} > \mathsf{U}, \mathsf{W}, \mathsf{R} > \wedge, \vee > \rightarrow$
  - omit outer parentheses
  - $\rightarrow, \wedge, \vee$ are right–associative

Example

formula $\mathsf{F}(p \to \mathsf{G}\, r) \vee \neg q\, \mathsf{U}\, p$ $\qquad$ $\mathsf{F}\, p \to (\mathsf{G}\, r \vee \neg q)\, \mathsf{U}\, p$

parse tree





X	next state	F	$\exists$ future state	W	weak until
U	until	G	$\forall$ states globally	R	release

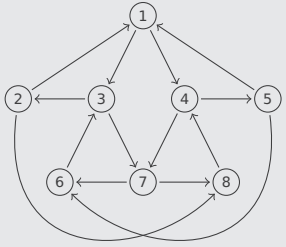
Outline

Definition

transition system (model) is triple $\mathcal{M} = (S, \to, L)$ with

1. set of states S
2. transition relation $\to \subseteq S \times S$ such that $\forall s \in S \;\; \exists t \in S$ with $s \to t$ ("no deadlock")
3. labeling function $L\colon S \to 2^{\text{atoms}}$

Example



model $\mathcal{M} = (S, \to, L)$

$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$L(1) = \{I_A, I_B\}$ $\quad$ $L(5) = \{I_A, P_B\}$

$L(2) = \{P_A, I_B\}$ $\quad$ $L(6) = \{R_A, P_B\}$

$L(3) = \{R_A, I_B\}$ $\quad$ $L(7) = \{R_A, R_B\}$

$L(4) = \{I_A, R_B\}$ $\quad$ $L(8) = \{P_A, R_B\}$

Definition

- path in model $\mathcal{M} = (S, \to, L)$ is infinite sequence $s_1 \to s_2 \to \cdots$
- $\forall$ paths $\pi = s_1 \to s_2 \to \cdots \quad \forall\, i \geqslant 1 \quad \pi^i = s_i \to s_{i+1} \to \cdots$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

$$\pi \vDash \varphi$$

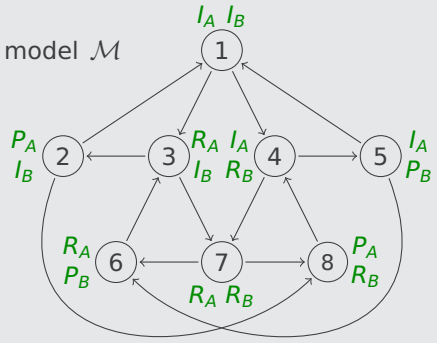
is defined by induction on φ:

$$\pi \vDash \top \qquad \pi \nvDash \bot \qquad \pi \vDash \varphi \wedge \psi \iff \pi \vDash \varphi \text{ and } \pi \vDash \psi$$

$$\pi \vDash p \iff p \in L(s_1) \qquad \pi \vDash \varphi \vee \psi \iff \pi \vDash \varphi \text{ or } \pi \vDash \psi$$

$$\pi \vDash \neg\varphi \iff \pi \nvDash \varphi \qquad \pi \vDash \varphi \to \psi \iff \pi \nvDash \varphi \text{ or } \pi \vDash \psi$$

Example



$$\pi_1 = 1 \to 3 \to 2 \to 1 \to 3 \to 2 \to \cdots = (1\,3\,2)^{\omega}$$

$$\pi_2 = 7 \to 6 \to 3 \to 7 \to 6 \to 3 \to \cdots = (7\,6\,3)^{\omega}$$

$\pi_1 \models I_A$ $\quad$ $\pi_1 \not\models R_A \wedge I_B$ $\quad$ $\pi_1 \not\models I_B \to P_A \vee R_B$

$\pi_2 \not\models I_A$ $\quad$ $\pi_2^6 \models R_A \wedge I_B$ $\quad$ $\pi_2 \models I_B \to P_A \vee R_B$

Definition

satisfaction of LTL formula φ with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$

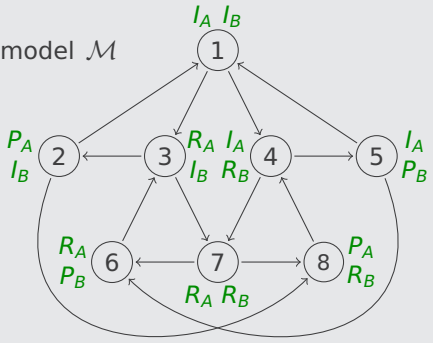
$$\pi \vDash \varphi$$

is defined by induction on φ:

$$\begin{aligned}
\pi &\vDash \mathsf{X}\varphi && \iff && \pi^2 \vDash \varphi \\
\pi &\vDash \mathsf{F}\varphi && \iff && \exists\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi &\vDash \mathsf{G}\varphi && \iff && \forall\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi &\vDash \varphi \mathsf{U} \psi && \iff && \exists\, i \geqslant 1 \quad \pi^i \vDash \psi \quad \text{and} \quad \forall\, j < i \quad \pi^j \vDash \varphi \\
\pi &\vDash \varphi \mathsf{W} \psi && \iff && \big(\exists\, i \geqslant 1 \quad \pi^i \vDash \psi \quad \text{and} \quad \forall\, j < i \quad \pi^j \vDash \varphi\big) \quad \text{or} \quad \forall\, i \geqslant 1 \quad \pi^i \vDash \varphi \\
\pi &\vDash \varphi \mathsf{R} \psi && \iff && \big(\exists\, i \geqslant 1 \quad \pi^i \vDash \varphi \quad \text{and} \quad \forall\, j \leqslant i \quad \pi^j \vDash \psi\big) \quad \text{or} \quad \forall\, i \geqslant 1 \quad \pi^i \vDash \psi
\end{aligned}$$

Example

model $\mathcal{M}$



$\pi_1 = (1\,3\,2)^\omega \qquad \pi_2 = (7\,6\,3)^\omega$

$\pi_1 \models \mathsf{X}(R_A \vee R_B)$

$\pi_2 \not\models \mathsf{F}\, P_A$

$\pi_1 \not\models I_A \,\mathsf{U}\, P_A$

$\pi_1 \models \mathsf{F}\, P_A$

$\pi_2 \models \mathsf{G} \neg I_A$

$\pi_2 \models \neg I_A \,\mathsf{W}\, P_A$

$\pi_1 \not\models \mathsf{X}\,\mathsf{X}\, P_B$

$\pi_2 \models \mathsf{G}\,\mathsf{F}\, P_B$

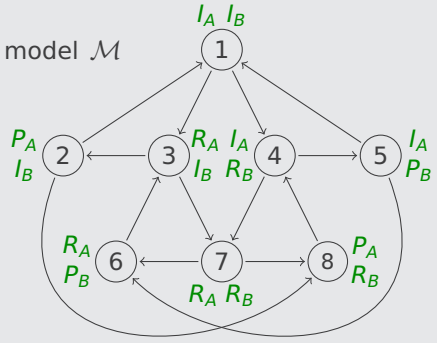
$\pi_2 \not\models P_B \,\mathsf{R}\, R_B$

Definition

model $\mathcal{M} = (S, \to, L)$, state $s \in S$, LTL formula φ

$\mathcal{M}, s \vDash \varphi \iff \forall$ paths $\pi = s \to \cdots \quad \pi \vDash \varphi$ "formula φ holds in state s of model $\mathcal{M}$"

Example



$\mathcal{M}, 1 \nvDash \mathrm{G}(R_A \to \mathrm{F}\, P_A)$

$\mathcal{M}, 4 \nvDash \neg(R_B \,\mathrm{U}\, P_B)$

$\mathcal{M}, 4 \nvDash \quad R_B \,\mathrm{U}\, P_B$

$\mathcal{M}, 6 \vDash \mathrm{X}(\mathrm{F}\, I_B \wedge ((\mathrm{X} \neg P_B)\, \mathrm{R}\, R_A))$

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$\forall$ models $\mathcal{M} = (S, \to, L)$

$\forall$ paths π in $\mathcal{M}$

$$\pi \vDash \varphi \iff \pi \vDash \psi$$

Theorem

$$\neg \mathsf{X}\varphi \equiv \mathsf{X}\neg\varphi$$
$$\neg \mathsf{F}\varphi \equiv \mathsf{G}\neg\varphi$$
$$\neg \mathsf{G}\varphi \equiv \mathsf{F}\neg\varphi$$
$$\neg(\varphi \mathsf{U} \psi) \equiv \neg\varphi \mathsf{R} \neg\psi$$
$$\neg(\varphi \mathsf{R} \psi) \equiv \neg\varphi \mathsf{U} \neg\psi$$
$$\varphi \mathsf{U} \psi \equiv \varphi \mathsf{W} \psi \wedge \mathsf{F}\psi$$
$$\varphi \mathsf{W} \psi \equiv \varphi \mathsf{U} \psi \vee \mathsf{G}\varphi$$

$$\varphi \mathsf{U} \psi \equiv \neg(\neg\psi \mathsf{U} (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\psi$$
$$\mathsf{F}(\varphi \vee \psi) \equiv \mathsf{F}\varphi \vee \mathsf{F}\psi$$
$$\mathsf{G}(\varphi \wedge \psi) \equiv \mathsf{G}\varphi \wedge \mathsf{G}\psi$$
$$\mathsf{F}\varphi \equiv \top \mathsf{U} \varphi$$
$$\mathsf{G}\varphi \equiv \bot \mathsf{R} \varphi$$
$$\varphi \mathsf{W} \psi \equiv \psi \mathsf{R} (\varphi \vee \psi)$$
$$\varphi \mathsf{R} \psi \equiv \psi \mathsf{W} (\varphi \wedge \psi)$$

Theorem

$\varphi \,\mathsf{U}\, \psi \equiv \neg(\neg\psi \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$

see overlay version for proof

Outline

Theorem

$\{X, U\}$, $\{X, W\}$ and $\{X, R\}$ are adequate sets of temporal connectives for LTL

Proof

- $F\varphi \equiv \top U \varphi$
- $G\varphi \equiv \neg F \neg \varphi$
- $\varphi R \psi \equiv \neg(\neg\varphi U \neg\psi)$
- $\varphi W \psi \equiv \varphi U \psi \vee G\varphi$

- $\varphi R \psi \equiv \psi W (\varphi \wedge \psi)$
- $\varphi U \psi \equiv \neg(\neg\varphi R \neg\psi)$
- $F\varphi \equiv \top U \varphi$
- $G\varphi \equiv \neg F \neg \varphi$
- $\varphi W \psi \equiv \varphi U \psi \vee G\varphi$

- $\varphi R \psi \equiv \psi W (\varphi \wedge \psi)$
- $\varphi U \psi \equiv \neg(\neg\varphi R \neg\psi)$
- $F\varphi \equiv \top U \varphi$
- $G\varphi \equiv \neg F \neg \varphi$
- $\varphi W \psi \equiv \varphi U \psi \vee G\varphi$

Outline

Esparza and Blondin

- Sections 11.1 and 13.2 of Automata Theory: An Algorithmic Approach (MIT Press 2023)

Important Concepts

- adequacy
- generalized Büchi automaton
- F
- G
- GBA
- linear–time temporal logic
- LTL
- R
- U
- W
- X

homework for January 9