universität innsbruck

Automata and Logic

Aart Middeldorp and Samuel Frontull

Outline

1. **Summary of Previous Lecture**
2. **LTL Model Checking**
3. **Intermezzo**
4. **LTL Model Checking**
5. **Further Reading**

Definition

- generalized Büchi automaton (GBA) is automaton $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$ with $F_1, \dots, F_k \subseteq Q$
- $\infty(r) = \{q \in Q \mid q \text{ occurs infinitely often in run } r\}$
- run r of GBA is accepting if $\infty(r) \cap F_i \neq \varnothing$ for all $1 \leqslant i \leqslant k$

Remark

NBA $(Q, \Sigma, \Delta, S, F)$ is equivalent to GBA $(Q, \Sigma, \Delta, S, \{F\})$

Lemma

every GBA can be transformed into equivalent NBA

Remark

minimal DBAs are not unique

Definition

LTL (linear–time temporal logic) formulas are built from propositional atoms, logical connectives and temporal connectives X, F, G, U, W, R, according to following grammar:

$$\varphi \ ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathsf{X}\,\varphi \mid \mathsf{F}\,\varphi \mid \mathsf{G}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \varphi\,\mathsf{W}\,\varphi \mid \varphi\,\mathsf{R}\,\varphi$$

Definitions

- transition system (model) is triple $\mathcal{M} = (S, \rightarrow, L)$ with
  1. set of states S
  2. transition relation $\rightarrow \subseteq S \times S$ such that $\forall s \in S \ \exists t \in S$ with $s \rightarrow t$ ("no deadlock")
  3. labeling function $L \colon S \rightarrow 2^{\text{atoms}}$
- path in model $\mathcal{M}$ is infinite sequence $s_1 \rightarrow s_2 \rightarrow \cdots$
- $\forall$ paths $\pi = s_1 \rightarrow s_2 \rightarrow \cdots \quad \forall i \geqslant 1 \quad \pi^i = s_i \rightarrow s_{i+1} \rightarrow \cdots$

Definition

satisfaction of LTL formula φ ($\pi \vDash \varphi$) with respect to path $\pi = s_1 \to s_2 \to \cdots$ in model $\mathcal{M} = (S, \to, L)$ is defined by induction on φ:

$$\pi \vDash \top \qquad \pi \nvDash \bot \qquad \pi \vDash \varphi \wedge \psi \iff \pi \vDash \varphi \text{ and } \pi \vDash \psi$$

$$\pi \vDash p \iff p \in L(s_1) \qquad \pi \vDash \varphi \vee \psi \iff \pi \vDash \varphi \text{ or } \pi \vDash \psi$$

$$\pi \vDash \neg\varphi \iff \pi \nvDash \varphi \qquad \pi \vDash \varphi \to \psi \iff \pi \nvDash \varphi \text{ or } \pi \vDash \psi$$

$$\pi \vDash \mathsf{X}\,\varphi \iff \pi^2 \vDash \varphi \qquad \pi \vDash \mathsf{F}\,\varphi \iff \exists\, i \geqslant 1 \;\; \pi^i \vDash \varphi$$

$$\pi \vDash \mathsf{G}\,\varphi \iff \forall\, i \geqslant 1 \;\; \pi^i \vDash \varphi$$

$$\pi \vDash \varphi\,\mathsf{U}\,\psi \iff \exists\, i \geqslant 1 \;\; \pi^i \vDash \psi \text{ and } \forall\, j < i \;\; \pi^j \vDash \varphi$$

$$\pi \vDash \varphi\,\mathsf{W}\,\psi \iff (\exists\, i \geqslant 1 \;\; \pi^i \vDash \psi \text{ and } \forall\, j < i \;\; \pi^j \vDash \varphi) \text{ or } \forall\, i \geqslant 1 \;\; \pi^i \vDash \varphi$$

$$\pi \vDash \varphi\,\mathsf{R}\,\psi \iff (\exists\, i \geqslant 1 \;\; \pi^i \vDash \varphi \text{ and } \forall\, j \leqslant i \;\; \pi^j \vDash \psi) \text{ or } \forall\, i \geqslant 1 \;\; \pi^i \vDash \psi$$

Definitions

- model $\mathcal{M} = (S, \rightarrow, L)$, state $s \in S$, LTL formula φ

$$\mathcal{M}, s \vDash \varphi \iff \forall \text{ paths } \pi = s \rightarrow \cdots \quad \pi \vDash \varphi \qquad \text{"formula } \varphi \text{ holds in state } s \text{ of } \mathcal{M}\text{"}$$

- LTL formulas φ and ψ are semantically equivalent $(\varphi \equiv \psi)$ if $\quad \pi \vDash \varphi \iff \pi \vDash \psi$ for all models $\mathcal{M} = (S, \rightarrow, L)$ and paths π in $\mathcal{M}$

Theorem

-

$$\begin{array}{ccc}
\neg\, \mathsf{X}\, \varphi \equiv \mathsf{X} \neg \varphi & \varphi\, \mathsf{U}\, \psi \equiv \neg(\neg \psi\, \mathsf{U}\, (\neg \varphi \wedge \neg \psi)) \wedge \mathsf{F}\, \psi & \\
\neg\, \mathsf{F}\, \varphi \equiv \mathsf{G} \neg \varphi & \mathsf{F}\, \varphi \equiv \top\, \mathsf{U}\, \varphi & \mathsf{F}(\varphi \vee \psi) \equiv \mathsf{F}\, \varphi \vee \mathsf{F}\, \psi \\
\neg\, \mathsf{G}\, \varphi \equiv \mathsf{F} \neg \varphi & \mathsf{G}\, \varphi \equiv \bot\, \mathsf{R}\, \varphi & \mathsf{G}(\varphi \wedge \psi) \equiv \mathsf{G}\, \varphi \wedge \mathsf{G}\, \psi \\
\neg(\varphi\, \mathsf{U}\, \psi) \equiv \neg \varphi\, \mathsf{R} \neg \psi & \varphi\, \mathsf{W}\, \psi \equiv \psi\, \mathsf{R}\, (\varphi \vee \psi) & \varphi\, \mathsf{U}\, \psi \equiv \varphi\, \mathsf{W}\, \psi \wedge \mathsf{F}\, \psi \\
\neg(\varphi\, \mathsf{R}\, \psi) \equiv \neg \varphi\, \mathsf{U} \neg \psi & \varphi\, \mathsf{R}\, \psi \equiv \psi\, \mathsf{W}\, (\varphi \wedge \psi) & \varphi\, \mathsf{W}\, \psi \equiv \varphi\, \mathsf{U}\, \psi \vee \mathsf{G}\, \varphi
\end{array}$$

- $\{\mathsf{X}, \mathsf{U}\}$, $\{\mathsf{X}, \mathsf{W}\}$ and $\{\mathsf{X}, \mathsf{R}\}$ are adequate sets of temporal connectives for LTL

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

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① construct GBA $A_{\neg\varphi}$ for $\neg\varphi$

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AP is (finite) set of propositional atoms used in $\mathcal{M}$ and φ

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Plan

GBA $A_{\neg\varphi}$ accepts all traces that violate φ

formula φ in LTL fragment with U and X as only temporal operators

Definitions

- $L(\varphi) = \{x \in (2^{\mathsf{AP}})^\omega \mid x \vDash \varphi\}$

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- $L(\varphi) = \{x \in (2^{\mathrm{AP}})^\omega \mid x \vDash \varphi\}$
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$$x \vDash \top$$

formula φ in LTL fragment with U and X as only temporal operators

Definitions

- $L(\varphi) = \{x \in (2^{\mathrm{AP}})^\omega \mid x \models \varphi\}$
- $x \models \varphi$ is defined inductively $\quad (x = x_0 x_1 x_2 \cdots)$

$$x \models \top$$

$$x \models p \qquad \iff \qquad p \in x_0$$

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\end{array}
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Plan

GBA A_φ will accept $L(\varphi)$

Basic Idea

- given infinite string $x = x_0 x_1 x_2 \cdots \in L(\varphi)$

  expand $x_i \subseteq \mathrm{AP}$ by subformulas ψ of φ to obtain infinite string $y = y_0 y_1 y_2 \cdots$ such that

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Example

$\varphi = a \,\mathsf{U}\, (\neg a \wedge b)$ and $x = \{a\}\{a, b\}\{b\} \cdots$

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- $y_0,\ y_1,\ \ldots$ are states in A_φ

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- $y_0,\ y_1,\ \ldots$ are states in A_φ
- transition relation and acceptance conditions of A_φ ensure that y is accepting run for x

Example

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③ maximal: for all $\psi \in \mathcal{C}(\varphi)$

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Example

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- $\{a, b, \neg a \wedge b, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
- $\{a, b, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
- $\{a, b, \neg(\neg a \wedge b), a \,\mathsf{U}\, (\neg a \wedge b)\}$

Definition

closure $\mathcal{C}(\varphi)$ of φ consists of all subformulas of φ and their negations, identifying $\neg\neg\psi$ and ψ

Example

$\mathcal{C}(a \,\mathsf{U}\, (\neg a \wedge b)) = \{a, b, \neg a, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), a \,\mathsf{U}\, (\neg a \wedge b), \neg(a \,\mathsf{U}\, (\neg a \wedge b))\}$

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- $\{a, b, \neg a \wedge b, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
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- $\{a, b, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
- $\{a, b, \neg(\neg a \wedge b), a \,\mathsf{U}\, (\neg a \wedge b)\}$ elementary
- $\{\neg a, \neg b, \neg(\neg a \wedge b), a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary

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- $\{a, b, \neg a \wedge b, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
- $\{a, b, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
- $\{a, b, \neg(\neg a \wedge b), a \,\mathsf{U}\, (\neg a \wedge b)\}$ elementary
- $\{\neg a, \neg b, \neg(\neg a \wedge b), a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
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$\mathcal{C}(a \,\mathsf{U}\, (\neg a \wedge b)) = \{a,\, b,\, \neg a,\, \neg b,\, \neg a \wedge b,\, \neg(\neg a \wedge b),\, a \,\mathsf{U}\, (\neg a \wedge b),\, \neg(a \,\mathsf{U}\, (\neg a \wedge b))\}$

- $\{a,\, b,\, \neg a \wedge b,\, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
- $\{a,\, b,\, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
- $\{a,\, b,\, \neg(\neg a \wedge b),\, a \,\mathsf{U}\, (\neg a \wedge b)\}$ elementary
- $\{\neg a,\, \neg b,\, \neg(\neg a \wedge b),\, a \,\mathsf{U}\, (\neg a \wedge b)\}$ not elementary
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Example

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Lemma

$$\varphi \, \mathsf{U} \, \psi \;\equiv\; \psi \;\vee\; (\varphi \;\wedge\; \mathsf{X}(\varphi \, \mathsf{U} \, \psi))$$

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Definition

GBA $A_\varphi = (Q, 2^{\mathsf{AP}}, \Delta, S, F)$ for LTL formula φ with atoms in AP

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- $S = \{B \in Q \mid \varphi \in B\}$

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- $\Delta(B, A) = \begin{cases} \varnothing & \text{if } A \neq B \cap \mathsf{AP} \\ \end{cases}$

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- $S = \{B \in Q \mid \varphi \in B\}$
- $\Delta(B, A) = \begin{cases} \varnothing & \text{if } A \neq B \cap \mathrm{AP} \\ \{C \mid C \in Q \text{ and } \dots\} & \text{if } A = B \cap \mathrm{AP} \end{cases}$

Lemma

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  ① for all $\mathsf{X}\,\psi \in \mathcal{C}(\varphi)$ $\quad \mathsf{X}\,\psi \in B \iff \psi \in C$

Lemma

$\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \lor (\varphi \land \mathsf{X}(\varphi \,\mathsf{U}\, \psi))$

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- $Q = \{B \subseteq \mathcal{C}(\varphi) \mid B \text{ is elementary}\}$
- $S = \{B \in Q \mid \varphi \in B\}$
- $\Delta(B, A) = \begin{cases} \varnothing & \text{if } A \neq B \cap \mathsf{AP} \\ \{C \mid C \in Q \text{ and } \ldots\} & \text{if } A = B \cap \mathsf{AP} \end{cases}$ with

  ① for all $\mathsf{X}\,\psi \in \mathcal{C}(\varphi)$ $\quad \mathsf{X}\,\psi \in B \iff \psi \in C$

  ② for all $\varphi_1 \,\mathsf{U}\, \varphi_2 \in \mathcal{C}(\varphi)$ $\quad \varphi_1 \,\mathsf{U}\, \varphi_2 \in B \iff \varphi_2 \in B$ or both $\varphi_1 \in B$ and $\varphi_1 \,\mathsf{U}\, \varphi_2 \in C$

Lemma

$\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \;\vee\; (\varphi \;\wedge\; \mathsf{X}(\varphi \,\mathsf{U}\, \psi))$

Definition

GBA $A_\varphi = (Q, 2^{\mathrm{AP}}, \Delta, S, F)$ for LTL formula φ with atoms in AP:

- $Q = \{B \subseteq \mathcal{C}(\varphi) \mid B \text{ is elementary}\}$
- $S = \{B \in Q \mid \varphi \in B\}$
- $\Delta(B, A) = \begin{cases} \varnothing & \text{if } A \neq B \cap \mathrm{AP} \\ \{C \mid C \in Q \text{ and } \ldots\} & \text{if } A = B \cap \mathrm{AP} \end{cases}$ with

  ① for all $\mathsf{X}\,\psi \in \mathcal{C}(\varphi)$ $\quad \mathsf{X}\,\psi \in B \iff \psi \in C$

  ② for all $\varphi_1 \,\mathsf{U}\, \varphi_2 \in \mathcal{C}(\varphi)$ $\quad \varphi_1 \,\mathsf{U}\, \varphi_2 \in B \iff \varphi_2 \in B$ or both $\varphi_1 \in B$ and $\varphi_1 \,\mathsf{U}\, \varphi_2 \in C$
- $F = \{\{B \in Q \mid \varphi_1 \,\mathsf{U}\, \varphi_2 \notin B \text{ or } \varphi_2 \in B\} \mid \varphi_1 \,\mathsf{U}\, \varphi_2 \in \mathcal{C}(\varphi)\}$

Example

$\varphi = \mathsf{X}\, a$

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- $\mathcal{C}(\varphi) = \{a, \neg a, \mathsf{X}\, a, \neg\, \mathsf{X}\, a\}$

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- $\mathcal{C}(\varphi) = \{a, \neg a, \mathsf{X}\, a, \neg \mathsf{X}\, a\}$
- states ❶ $\{a, \mathsf{X}\, a\}$ ❷ $\{a, \neg \mathsf{X}\, a\}$ ❸ $\{\neg a, \mathsf{X}\, a\}$ ❹ $\{\neg a, \neg \mathsf{X}\, a\}$

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- start states ❶ ❸

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- start states ❶ ❸
- transitions

	❶	❷	❸	❹
❶	✓	✓		

$\{a\}$

Example

$\varphi = \mathsf{X}\, a$

- $\mathcal{C}(\varphi) = \{a, \neg a, \mathsf{X}\, a, \neg\, \mathsf{X}\, a\}$
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- start states ❶ ❸
- transitions

	❶	❷	❸	❹	
❶	✓	✓			$\{a\}$
❷			✓	✓	$\{a\}$

Example

$\varphi = \mathrm{X}\, a$

- $\mathcal{C}(\varphi) = \{a, \neg a, \mathrm{X}\, a, \neg \mathrm{X}\, a\}$
- states ❶ $\{a, \mathrm{X}\, a\}$ ❷ $\{a, \neg \mathrm{X}\, a\}$ ❸ $\{\neg a, \mathrm{X}\, a\}$ ❹ $\{\neg a, \neg \mathrm{X}\, a\}$
- start states ❶ ❸
- transitions

	❶	❷	❸	❹	
❶	✓	✓			$\{a\}$
❷			✓	✓	$\{a\}$
❸	✓	✓			$\varnothing$

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- start states ❶ ❸
- transitions

	❶	❷	❸	❹	
❶	✓	✓			$\{a\}$
❷			✓	✓	$\{a\}$
❸	✓	✓			$\varnothing$
❹			✓	✓	$\varnothing$

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- start states ❶ ❸
- transitions

	❶	❷	❸	❹	
❶	✓	✓			$\{a\}$
❷			✓	✓	$\{a\}$
❸	✓	✓			$\varnothing$
❹			✓	✓	$\varnothing$

- acceptance condition $\varnothing \implies$ all runs are accepting

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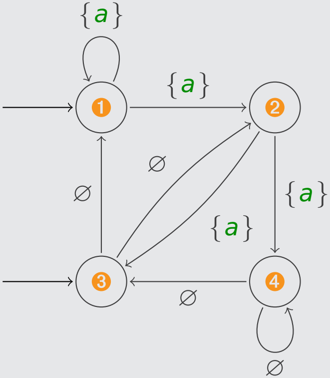
- states ❶ $\{a, \mathsf{X}\, a\}$ ❷ $\{a, \neg\, \mathsf{X}\, a\}$ ❸ $\{\neg a, \mathsf{X}\, a\}$ ❹ $\{\neg a, \neg\, \mathsf{X}\, a\}$
- start states ❶ ❸
- transitions

	❶	❷	❸	❹	
❶	✓	✓			$\{a\}$
❷			✓	✓	$\{a\}$
❸	✓	✓			$\varnothing$
❹			✓	✓	$\varnothing$

- acceptance condition $\varnothing \implies$ all runs are accepting



Example

$\varphi = a \, \mathsf{U} \, b$

Example

$\varphi = a \, \mathsf{U} \, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \, \mathsf{U} \, b, \neg(a \, \mathsf{U} \, b)\}$

Example

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a,\ \neg a,\ b,\ \neg b,\ a \,\mathsf{U}\, b,\ \neg(a \,\mathsf{U}\, b)\}$
- states ❶ $\{a,\ b,\ \varphi\}$ ❷ $\{\neg a,\ b,\ \varphi\}$ ❸ $\{a,\ \neg b,\ \varphi\}$ ❹ $\{a,\ \neg b,\ \neg\varphi\}$ ❺ $\{\neg a,\ \neg b,\ \neg\varphi\}$

Example

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a,\ \neg a,\ b,\ \neg b,\ a \,\mathsf{U}\, b,\ \neg(a \,\mathsf{U}\, b)\}$
- states ❶ $\{a,\ b,\ \varphi\}$ ❷ $\{\neg a,\ b,\ \varphi\}$ ❸ $\{a,\ \neg b,\ \varphi\}$ ❹ $\{a,\ \neg b,\ \neg\varphi\}$ ❺ $\{\neg a,\ \neg b,\ \neg\varphi\}$
- start states ❶ ❷ ❸

Example

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ❶ $\{a, b, \varphi\}$ ❷ $\{\neg a, b, \varphi\}$ ❸ $\{a, \neg b, \varphi\}$ ❹ $\{a, \neg b, \neg\varphi\}$ ❺ $\{\neg a, \neg b, \neg\varphi\}$
- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺
❶	✓	✓	✓	✓	✓

$\{a, b\}$

Example

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ❶ $\{a, b, \varphi\}$ ❷ $\{\neg a, b, \varphi\}$ ❸ $\{a, \neg b, \varphi\}$ ❹ $\{a, \neg b, \neg\varphi\}$ ❺ $\{\neg a, \neg b, \neg\varphi\}$
- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$

Example

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
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- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$

Example

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- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$
❹				✓	✓	$\{a\}$

Example

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ❶ $\{a, b, \varphi\}$ ❷ $\{\neg a, b, \varphi\}$ ❸ $\{a, \neg b, \varphi\}$ ❹ $\{a, \neg b, \neg\varphi\}$ ❺ $\{\neg a, \neg b, \neg\varphi\}$
- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$
❹				✓	✓	$\{a\}$
❺	✓	✓	✓	✓	✓	$\varnothing$

Example

$\varphi = a \, \mathsf{U} \, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \, \mathsf{U} \, b, \neg(a \, \mathsf{U} \, b)\}$
- states ❶ $\{a, b, \varphi\}$ ❷ $\{\neg a, b, \varphi\}$ ❸ $\{a, \neg b, \varphi\}$ ❹ $\{a, \neg b, \neg\varphi\}$ ❺ $\{\neg a, \neg b, \neg\varphi\}$
- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$
❹				✓	✓	$\{a\}$
❺	✓	✓	✓	✓	✓	$\varnothing$

- acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

Example

$\varphi = a \,\mathsf{U}\, b$

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- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$
❹				✓	✓	$\{a\}$
❺	✓	✓	✓	✓	✓	$\varnothing$

- acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$ $\Longrightarrow$ runs cycling in state ❸ are not accepting

Example

$\varphi = a \mathbin{\mathsf{U}} b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \mathbin{\mathsf{U}} b, \neg(a \mathbin{\mathsf{U}} b)\}$
- states ❶ $\{a, b, \varphi\}$ ❷ $\{\neg a, b, \varphi\}$ ❸ $\{a, \neg b, \varphi\}$ ❹ $\{a, \neg b, \neg\varphi\}$ ❺ $\{\neg a, \neg b, \neg\varphi\}$
- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$
❹				✓	✓	$\{a\}$
❺	✓	✓	✓	✓	✓	$\varnothing$

- acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$ $\Longrightarrow$ runs cycling in state ❸ are not accepting
- $\{a\}^\omega \notin L(A_\varphi)$

Example

$\varphi = a \,\mathsf{U}\, b$

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- start states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$
❹				✓	✓	$\{a\}$
❺	✓	✓	✓	✓	✓	$\varnothing$

- acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$ $\Longrightarrow$ runs cycling in state ❸ are not accepting
- $\{a\}^\omega \notin L(A_\varphi)$ and $\{b\}\varnothing\{a\}^\omega \in L(A_\varphi)$

Theorem

$L(\varphi) \subseteq L(A_\varphi)$

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Proof

- GBA $A_\varphi = (Q, 2^{\mathrm{AP}}, \Delta, S, F)$

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- let $x = x_0 x_1 x_2 \cdots \in L(\varphi)$ and define $y_i = \{\psi \in \mathcal{C}(\varphi) \mid x_i x_{i+1} \cdots \vDash \psi\}$ for all $i \geqslant 0$

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- GBA $A_\varphi = (Q, 2^{\mathrm{AP}}, \Delta, S, F)$
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- $y_0, y_1, y_2, \cdots \in Q$

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- $y_0, y_1, y_2, \cdots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

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  $y_{i+1} \in \Delta(y_i, x_i)$

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- $y_0, y_1, y_2, \cdots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

  $y_{i+1} \in \Delta(y_i, x_i)$: $\quad x_i = y_i \cap \mathrm{AP}$

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- $y_0, y_1, y_2, \cdots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

  $y_{i+1} \in \Delta(y_i, x_i)$: $\quad x_i = y_i \cap \mathrm{AP}$ and for all $\mathsf{X}\,\psi \in \mathcal{C}(\varphi)$

$$\mathsf{X}\,\psi \in y_i \quad\Longleftrightarrow\quad x_i x_{i+1} \cdots \vDash \mathsf{X}\,\psi$$

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- let $x = x_0 x_1 x_2 \cdots \in L(\varphi)$ and define $y_i = \{\psi \in \mathcal{C}(\varphi) \mid x_i x_{i+1} \cdots \models \psi\}$ for all $i \geqslant 0$
- $y_0, y_1, y_2, \dots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

  $y_{i+1} \in \Delta(y_i, x_i)$: $\quad x_i = y_i \cap \mathrm{AP}$ and for all $\mathsf{X}\,\psi \in \mathcal{C}(\varphi)$

$$\mathsf{X}\,\psi \in y_i \quad\Longleftrightarrow\quad x_i x_{i+1} \cdots \models \mathsf{X}\,\psi \quad\Longleftrightarrow\quad x_{i+1} \cdots \models \psi$$

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$$\mathrm{X}\,\psi \in y_i \iff x_i x_{i+1} \cdots \vDash \mathrm{X}\,\psi \iff x_{i+1} \cdots \vDash \psi \iff \psi \in y_{i+1}$$

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  and for all $\varphi_1 \,\mathsf{U}\, \varphi_2 \in \mathcal{C}(\varphi)$

  $$\varphi_1 \,\mathsf{U}\, \varphi_2 \in y_i \iff x_i x_{i+1} \cdots \vDash \varphi_1 \,\mathsf{U}\, \varphi_2$$

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and for all $\varphi_1 \mathbin{\mathsf{U}} \varphi_2 \in \mathcal{C}(\varphi)$

$$\begin{aligned} \varphi_1 \mathbin{\mathsf{U}} \varphi_2 \in y_i &\iff x_i x_{i+1} \cdots \vDash \varphi_1 \mathbin{\mathsf{U}} \varphi_2 \\ &\iff x_i x_{i+1} \cdots \vDash \varphi_2 \text{ or } (x_i x_{i+1} \cdots \vDash \varphi_1 \text{ and } x_{i+1} \cdots \vDash \varphi_1 \mathbin{\mathsf{U}} \varphi_2) \end{aligned}$$

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  $$\mathrm{X}\,\psi \in y_i \iff x_i x_{i+1} \cdots \vDash \mathrm{X}\,\psi \iff x_{i+1} \cdots \vDash \psi \iff \psi \in y_{i+1}$$

  and for all $\varphi_1 \,\mathrm{U}\, \varphi_2 \in \mathcal{C}(\varphi)$

  $$\begin{aligned} \varphi_1 \,\mathrm{U}\, \varphi_2 \in y_i &\iff x_i x_{i+1} \cdots \vDash \varphi_1 \,\mathrm{U}\, \varphi_2 \\ &\iff x_i x_{i+1} \cdots \vDash \varphi_2 \text{ or } (x_i x_{i+1} \cdots \vDash \varphi_1 \text{ and } x_{i+1} \cdots \vDash \varphi_1 \,\mathrm{U}\, \varphi_2) \\ &\iff \varphi_2 \in y_i \text{ or } (\varphi_1 \in y_i \text{ and } \varphi_1 \,\mathrm{U}\, \varphi_2 \in y_{i+1}) \end{aligned}$$

Theorem

$L(\varphi) \subseteq L(A_\varphi)$

Proof (cont'd)

- GBA $A_\varphi = (Q, 2^{\mathrm{AP}}, \Delta, S, F)$
- let $x = x_0 x_1 x_2 \cdots \in L(\varphi)$ and define $y_i = \{\psi \in \mathcal{C}(\varphi) \mid x_i x_{i+1} \cdots \vDash \psi\}$ for all $i \geqslant 0$
- $y_0, y_1, y_2, \dots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

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  let $\varphi_1 \,\mathrm{U}\, \varphi_2 \in \mathcal{C}(\varphi)$

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  let $\varphi_1 \mathsf{U} \varphi_2 \in \mathcal{C}(\varphi)$ and suppose $y_i \notin \{B \in Q \mid \varphi_1 \mathsf{U} \varphi_2 \notin B$ or $\varphi_2 \in B\}$ for some i

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$$\varphi_1 \,\mathsf{U}\, \varphi_2 \in y_i \text{ and } \varphi_2 \notin y_i \quad\Longrightarrow\quad x_i x_{i+1} \cdots \vDash \varphi_1 \,\mathsf{U}\, \varphi_2 \text{ and } x_i x_{i+1} \cdots \nvDash \varphi_2$$

Theorem

$L(\varphi) \subseteq L(A_\varphi)$

Proof (cont'd)

- GBA $A_\varphi = (Q, 2^{AP}, \Delta, S, F)$
- let $x = x_0 x_1 x_2 \cdots \in L(\varphi)$ and define $y_i = \{\psi \in \mathcal{C}(\varphi) \mid x_i x_{i+1} \cdots \vDash \psi\}$ for all $i \geqslant 0$
- $y_0, y_1, y_2, \dots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

  let $\varphi_1 \mathbin{\mathsf{U}} \varphi_2 \in \mathcal{C}(\varphi)$ and suppose $y_i \notin \{B \in Q \mid \varphi_1 \mathbin{\mathsf{U}} \varphi_2 \notin B$ or $\varphi_2 \in B\}$ for some i

$$\begin{aligned} \varphi_1 \mathbin{\mathsf{U}} \varphi_2 \in y_i \text{ and } \varphi_2 \notin y_i \quad &\Longrightarrow \quad x_i x_{i+1} \cdots \vDash \varphi_1 \mathbin{\mathsf{U}} \varphi_2 \text{ and } x_i x_{i+1} \cdots \nvDash \varphi_2 \\ &\Longrightarrow \quad x_k x_{k+1} \cdots \vDash \varphi_2 \text{ for some } k > i \end{aligned}$$

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$L(\varphi) \subseteq L(A_\varphi)$

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- $y_0, y_1, y_2, \dots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

  let $\varphi_1 \,\mathsf{U}\, \varphi_2 \in \mathcal{C}(\varphi)$ and suppose $y_i \notin \{B \in Q \mid \varphi_1 \,\mathsf{U}\, \varphi_2 \notin B \text{ or } \varphi_2 \in B\}$ for some i

$$\begin{aligned}
\varphi_1 \,\mathsf{U}\, \varphi_2 \in y_i \text{ and } \varphi_2 \notin y_i \quad &\Longrightarrow \quad x_i x_{i+1} \cdots \vDash \varphi_1 \,\mathsf{U}\, \varphi_2 \text{ and } x_i x_{i+1} \cdots \nvDash \varphi_2 \\
&\Longrightarrow \quad x_k x_{k+1} \cdots \vDash \varphi_2 \text{ for some } k > i \\
&\Longrightarrow \quad \varphi_2 \in y_k \text{ for some } k > i
\end{aligned}$$

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- $y_0, y_1, y_2, \cdots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

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&\Longrightarrow \quad \varphi_2 \in y_k \text{ for some } k > i \\
&\Longrightarrow \quad y_k \in \{B \in Q \mid \varphi_1 \mathbin{\mathsf{U}} \varphi_2 \notin B \text{ or } \varphi_2 \in B\} \text{ for some } k > i
\end{aligned}
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$L(\varphi) \subseteq L(A_\varphi)$

Proof (cont'd)

- GBA $A_\varphi = (Q, 2^{AP}, \Delta, S, F)$
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- $y_0, y_1, y_2, \cdots \in Q$ and $y_0 y_1 y_2 \cdots$ is accepting run for x

  let $\varphi_1 \, \mathsf{U} \, \varphi_2 \in \mathcal{C}(\varphi)$ and suppose $y_i \notin \{B \in Q \mid \varphi_1 \, \mathsf{U} \, \varphi_2 \notin B$ or $\varphi_2 \in B\}$ for some i

$$
\begin{aligned}
\varphi_1 \, \mathsf{U} \, \varphi_2 \in y_i \text{ and } \varphi_2 \notin y_i \quad &\Longrightarrow \quad x_i x_{i+1} \cdots \vDash \varphi_1 \, \mathsf{U} \, \varphi_2 \text{ and } x_i x_{i+1} \cdots \nvDash \varphi_2 \\
&\Longrightarrow \quad x_k x_{k+1} \cdots \vDash \varphi_2 \text{ for some } k > i \\
&\Longrightarrow \quad \varphi_2 \in y_k \text{ for some } k > i \\
&\Longrightarrow \quad y_k \in \{B \in Q \mid \varphi_1 \, \mathsf{U} \, \varphi_2 \notin B \text{ or } \varphi_2 \in B\} \text{ for some } k > i
\end{aligned}
$$

  $y_k \in \{B \in Q \mid \varphi_1 \, \mathsf{U} \, \varphi_2 \notin B$ or $\varphi_2 \in B\}$ for infinitely many k

Theorem

$L(A_\varphi) \subseteq L(\varphi)$

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Proof

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- GBA $A_\varphi = (Q, 2^{\mathsf{AP}}, \Delta, S, F)$
- let $x = x_0 x_1 x_2 \cdots \in L(A_\varphi)$ with accepting run $y = y_0 y_1 y_2 \cdots$

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Proof

- GBA $A_\varphi = (Q, 2^{\mathrm{AP}}, \Delta, S, F)$
- let $x = x_0 x_1 x_2 \cdots \in L(A_\varphi)$ with accepting run $y = y_0 y_1 y_2 \cdots$
- $\Delta(B, A) = \varnothing$ if $A \neq B \cap \mathrm{AP} \quad \Longrightarrow \quad x = (y_0 \cap \mathrm{AP})(y_1 \cap \mathrm{AP})(y_2 \cap \mathrm{AP}) \cdots$

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- claim: $x \vDash \varphi$

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- GBA $A_\varphi = (Q, 2^{\mathrm{AP}}, \Delta, S, F)$
- let $x = x_0 x_1 x_2 \cdots \in L(A_\varphi)$ with accepting run $y = y_0 y_1 y_2 \cdots$
- $\Delta(B, A) = \varnothing$ if $A \neq B \cap \mathrm{AP} \quad \Longrightarrow \quad x = (y_0 \cap \mathrm{AP})(y_1 \cap \mathrm{AP})(y_2 \cap \mathrm{AP}) \cdots$
- claim: $x \vDash \varphi$

  more general claim: if y is "accepting run" (without requiring y_0 is start state) for x then

  $$\psi \in y_0 \quad \Longleftrightarrow \quad x \vDash \psi$$

  for all $\psi \in \mathcal{C}(\varphi)$

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- GBA $A_\varphi = (Q, 2^{\mathrm{AP}}, \Delta, S, F)$
- let $x = x_0 x_1 x_2 \cdots \in L(A_\varphi)$ with accepting run $y = y_0 y_1 y_2 \cdots$
- $\Delta(B, A) = \varnothing$ if $A \neq B \cap \mathrm{AP} \quad \Longrightarrow \quad x = (y_0 \cap \mathrm{AP})(y_1 \cap \mathrm{AP})(y_2 \cap \mathrm{AP}) \cdots$
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  for all $\psi \in \mathcal{C}(\varphi)$
- y_0 is start state $\quad \Longrightarrow \quad \varphi \in y_0$

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- GBA $A_\varphi = (Q, 2^{\mathsf{AP}}, \Delta, S, F)$
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- $\Delta(B, A) = \varnothing$ if $A \neq B \cap \mathsf{AP} \quad \Longrightarrow \quad x = (y_0 \cap \mathsf{AP})(y_1 \cap \mathsf{AP})(y_2 \cap \mathsf{AP}) \cdots$
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  $$\psi \in y_0 \quad \Longleftrightarrow \quad x \vDash \psi$$

  for all $\psi \in \mathcal{C}(\varphi)$
- y_0 is start state $\quad \Longrightarrow \quad \varphi \in y_0 \quad \Longrightarrow \quad x \vDash \varphi$

Outline

Question

Which of the following statements hold for $\varphi = (a \wedge \mathrm{X}\, b) \,\mathrm{U} \neg a$?

A $\neg \mathrm{X}\, a \in C(\varphi)$

B $\{b,\ a \wedge \mathrm{X}\, b,\ \varphi\}$ is locally consistent with respect to U

C $\{\neg a,\ b,\ \neg \mathrm{X}\, b,\ \neg(a \wedge \mathrm{X}\, b),\ \varphi\} \xrightarrow{\{b\}} \{a,\ \neg b,\ \mathrm{X}\, b,\ a \wedge \mathrm{X}\, b,\ \neg\varphi\}$ in A_φ

D $\{\neg a,\ b,\ \neg \mathrm{X}\, b,\ \neg(a \wedge \mathrm{X}\, b),\ \varphi\} \xleftarrow{\varnothing} \{a,\ b,\ \mathrm{X}\, b,\ a \wedge \mathrm{X}\, b,\ \neg\varphi\}$ in A_φ

Outline

1. **Summary of Previous Lecture**
2. **LTL Model Checking**
3. **Intermezzo**
4. **LTL Model Checking**
5. **Further Reading**

Remark

bit vectors can be used to represent 2^{AP}

Remark

bit vectors can be used to represent 2^{AP}: if $\mathrm{AP} = \{a, b, c\}$ then

$$\{a\}\varnothing\{a,c\}\{b,c\}\cdots = \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\cdots$$

Basic Strategy

$\mathcal{M}, s \vDash \varphi$?

① construct GBA $A_{\neg\varphi}$ for $\neg\varphi$

② combine $A_{\neg\varphi}$ and $\mathcal{M}, s$ into single automaton $A_{\neg\varphi} \times A_{\mathcal{M},s}$

③ test emptiness of $L(A_{\neg\varphi} \times A_{\mathcal{M},s})$

Basic Strategy

$\mathcal{M}, s \vDash \varphi$?

① construct GBA $A_{\neg\varphi}$ for $\neg\varphi$

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③ test emptiness of $L(A_{\neg\varphi} \times A_{\mathcal{M},s})$

Remark

model $\mathcal{M}$ is not GBA

Basic Strategy

$\mathcal{M}, s \vDash \varphi$?

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Remark

model $\mathcal{M}$ is not GBA

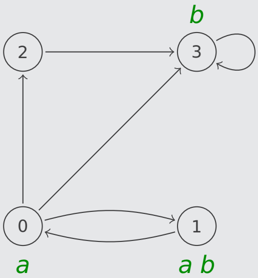
Definition

GBA $A_{\mathcal{M},s} = (S, 2^{\mathrm{AP}}, \Delta, \{s\}, \varnothing)$ for model $\mathcal{M} = (S, \to, L)$ and state $s \in S$

- $\Delta(p, A) = \{q \mid L(p) = A \text{ and } p \to q\}$

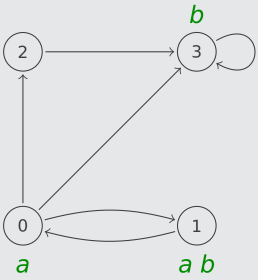
Example

model $\mathcal{M}$

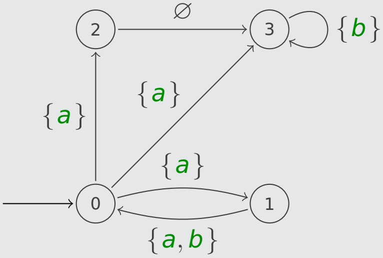


Example

model $\mathcal{M}$



GBA $A_{\mathcal{M},0}$



Intersection of GBAs

- $A = L(M_1)$ for GBA $M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ and $B = L(M_2)$ for GBA $M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$

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- $A \cap B = L(M)$ for GBA $M = (Q, \Sigma, \Delta, S, F)$ with
  1. $Q = Q_1 \times Q_2$
  2. $S = S_1 \times S_2$
  3. $\Delta((p,q),a) = \{(p',q') \mid p' \in \Delta_1(p,a) \text{ and } q' \in \Delta_2(q,a)\}$

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  4. $F = \{X \times Q_2 \mid X \in F_1\} \cup \{Q_1 \times Y \mid Y \in F_2\}$

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Theorem

problem

instance: GBA M

question: $L(M) = \varnothing$?

is decidable

Proof

① consider transition graph of GBA $M = (Q, \Sigma, \Delta, S, \{F_1, \dots, F_k\})$

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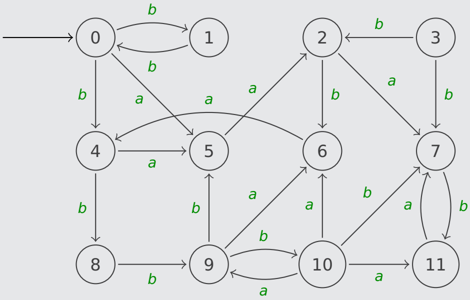
Example

- GBA M with acceptance sets $\{0, 9\}$, $\{0, 3, 10\}$, $\{5, 10, 11\}$



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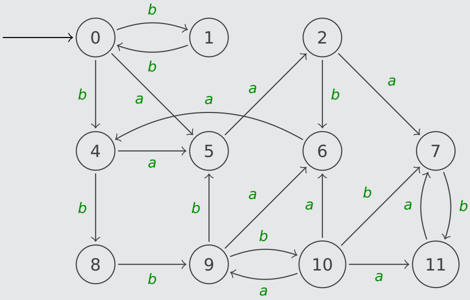

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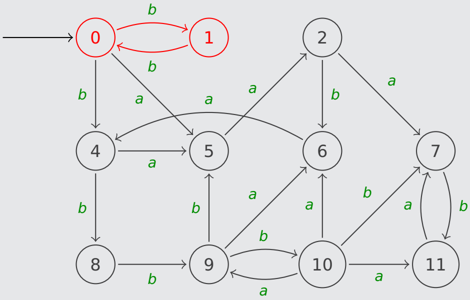
- three SCCs:

  ❶ $C_1 = \{0, 1\}$



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  1. $C_1 = \{0, 1\}$
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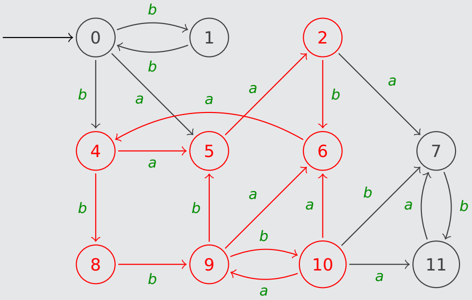

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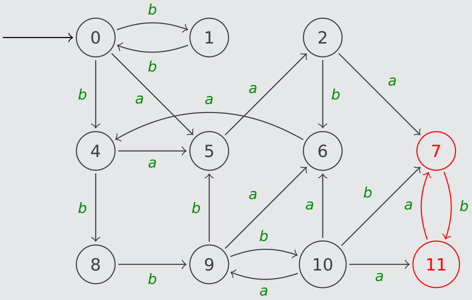

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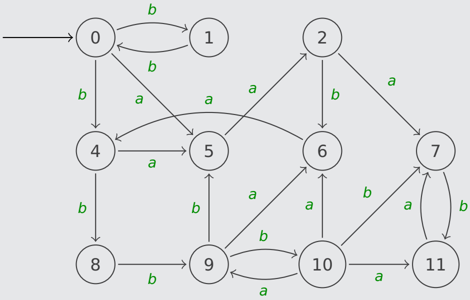

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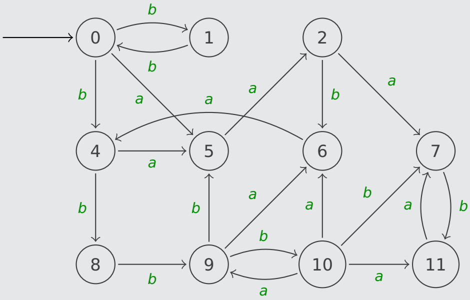

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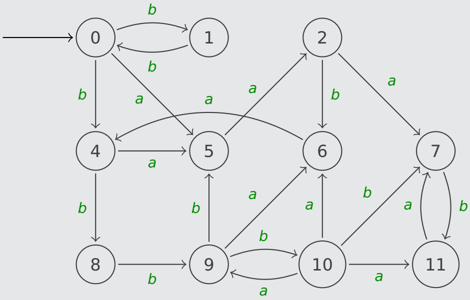

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  1. $C_1 = \{0,1\}$
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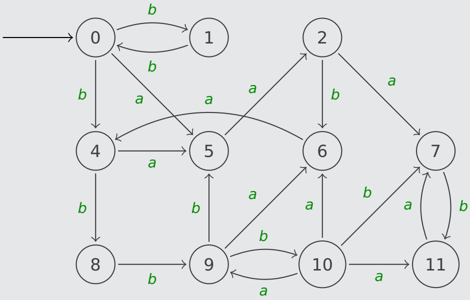
- $L(M) \neq \varnothing$



Basic Strategy

$\mathcal{M}, s \vDash \varphi$?

① construct GBA $A_{\neg\varphi}$ for $\neg\varphi$

② combine $A_{\neg\varphi}$ and $\mathcal{M}, s$ into single automaton $A_{\neg\varphi} \times A_{\mathcal{M},s}$

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Theorem

$$\mathcal{M}, s \vDash \varphi \quad \Longleftrightarrow \quad L(A_{\neg\varphi} \times A_{\mathcal{M},s}) = \varnothing$$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \, \mathsf{U} \, b$

		1	2	3	4	5
	$\{a, b, \varphi\}$ 1	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ 2	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ 3	✓	✓	✓		
$\to$	$\{a, \neg b, \neg\varphi\}$ 4				✓	✓
$\to$	$\{\neg a, \neg b, \neg\varphi\}$ 5	✓	✓	✓	✓	✓

acceptance condition $\{\{1, 2, 4, 5\}\}$

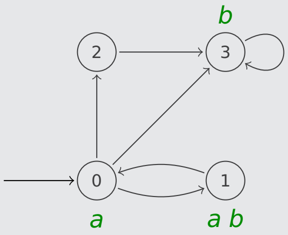
Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

		❶	❷	❸	❹	❺
	$\{a, b, \varphi\}$ ❶	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ ❷	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ ❸	✓	✓	✓		
$\rightarrow$	$\{a, \neg b, \neg\varphi\}$ ❹				✓	✓
$\rightarrow$	$\{\neg a, \neg b, \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

model $\mathcal{M}$



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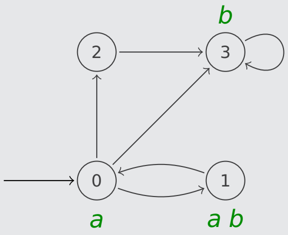
GBA $A_{\neg\varphi}$ for $\varphi = a \, \mathsf{U} \, b$

	❶	❷	❸	❹	❺
$\{a, b, \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ❸	✓	✓	✓		
$\to \{a, \neg b, \neg\varphi\}$ ❹				✓	✓
$\to \{\neg a, \neg b, \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

model $\mathcal{M}$



- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

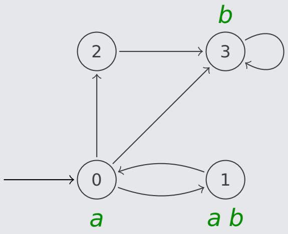
$\to$ ❹ 0	
$\to$ ❺ 0	

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \, \mathrm{U} \, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition $\{\{$①, ②, ④, ⑤$\}\}$

- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

$\rightarrow$ ④ 0	$\{$④ 1, ④ 2, ④ 3, ⑤ 1, ⑤ 2, ⑤ 3$\}$
$\rightarrow$ ⑤ 0	

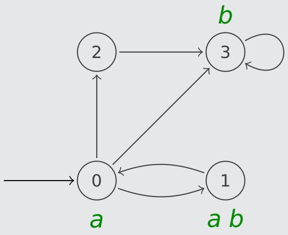
Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a, b, \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ❸	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ❹				✓	✓
$\rightarrow\{\neg a, \neg b, \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

model $\mathcal{M}$



- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

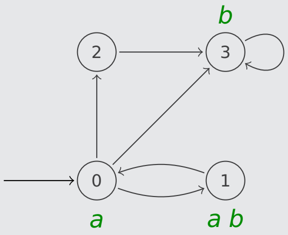
$\rightarrow$ ❹ 0	{❹ 1, ❹ 2, ❹ 3, ❺ 1, ❺ 2, ❺ 3}
$\rightarrow$ ❺ 0	$\varnothing$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathrm{U}\, b$

	❶	❷	❸	❹	❺
$\{a,\ b,\ \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a,\ b,\ \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a,\ \neg b,\ \varphi\}$ ❸	✓	✓	✓		
$\rightarrow$ $\{a,\ \neg b,\ \neg\varphi\}$ ❹				✓	✓
$\rightarrow \{\neg a,\ \neg b,\ \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

$\rightarrow$ ❹ 0	$\{$ ❹ 2, ❹ 3, ❺ 1, ❺ 2, ❺ 3$\}$
$\rightarrow$ ❺ 0	$\varnothing$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	1	2	3	4	5
$\{a, b, \varphi\}$ 1	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ 2	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ 3	✓	✓	✓		
$\rightarrow \{a, \neg b, \neg\varphi\}$ 4				✓	✓
$\rightarrow \{\neg a, \neg b, \neg\varphi\}$ 5	✓	✓	✓	✓	✓

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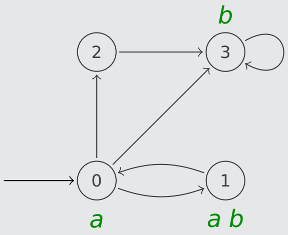


- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

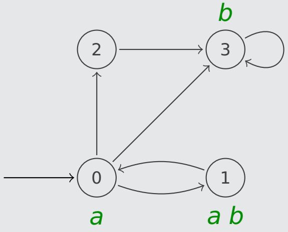
$\rightarrow$ 4 0	$\{$ 4 3, 5 1, 5 2, 5 3$\}$
$\rightarrow$ 5 0	$\varnothing$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a, b, \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ❸	✓	✓	✓		
$\rightarrow \{a, \neg b, \neg\varphi\}$ ❹				✓	✓
$\rightarrow \{\neg a, \neg b, \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

model $\mathcal{M}$



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$\rightarrow$ ❹ 0	$\{$ ❺ 1, ❺ 2, ❺ 3 $\}$
$\rightarrow$ ❺ 0	$\varnothing$

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GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a,\ b,\ \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a,\ b,\ \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a,\ \neg b,\ \varphi\}$ ❸	✓	✓	✓		
$\rightarrow\ \{a,\ \neg b,\ \neg\varphi\}$ ❹				✓	✓
$\rightarrow \{\neg a,\ \neg b,\ \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

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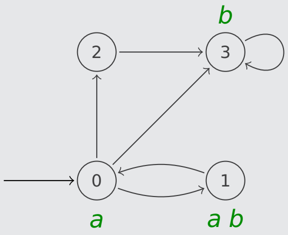

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$\rightarrow$ ❹ 0	$\{$ ❺ 2, ❺ 3 $\}$
$\rightarrow$ ❺ 0	$\varnothing$

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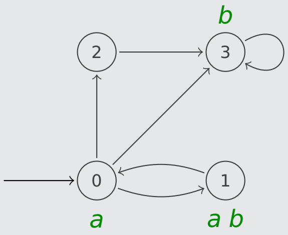
GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a, b, \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ❸	✓	✓	✓		
$\rightarrow \{a, \neg b, \neg\varphi\}$ ❹				✓	✓
$\rightarrow \{\neg a, \neg b, \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

model $\mathcal{M}$



- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

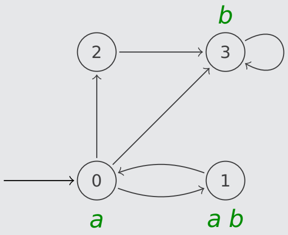
$\rightarrow$ ❹ 0	$\{$ ❺ 2 $\}$
$\rightarrow$ ❺ 0	$\varnothing$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a,\ b,\ \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a,\ b,\ \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a,\ \neg b,\ \varphi\}$ ❸	✓	✓	✓		
$\rightarrow$ $\{a,\ \neg b,\ \neg\varphi\}$ ❹				✓	✓
$\rightarrow$ $\{\neg a,\ \neg b,\ \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

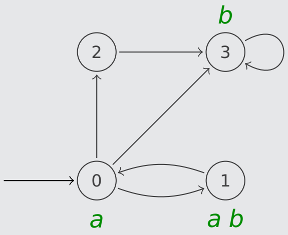
$\rightarrow$ ❹ 0	$\{$ ❺ 2 $\}$	❺ 2	$\{$❶ 3, ❷ 3, ❸ 3, ❹ 3, ❺ 3$\}$
$\rightarrow$ ❺ 0	$\varnothing$		

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a,\ b,\ \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a,\ b,\ \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a,\ \neg b,\ \varphi\}$ ❸	✓	✓	✓		
$\rightarrow$ $\{a,\ \neg b,\ \neg\varphi\}$ ❹				✓	✓
$\rightarrow$ $\{\neg a,\ \neg b,\ \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

$\rightarrow$ ❹ 0 | { ❺ 2 } ❺ 2 | { ❷ 3 }

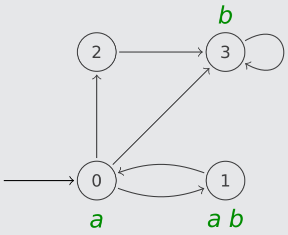
$\rightarrow$ ❺ 0 | $\varnothing$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a, b, \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ❸	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ❹				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

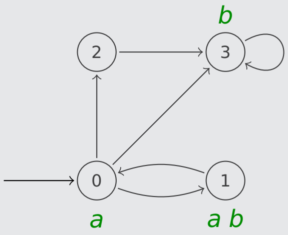
$\rightarrow$ ❹ 0	$\{$ ❺ 2 $\}$		❺ 2	$\{$ ❷ 3 $\}$
$\rightarrow$ ❺ 0	$\varnothing$		❷ 3	$\{$❶ 3, ❷ 3, ❸ 3, ❹ 3, ❺ 3$\}$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \, \mathsf{U} \, b$

	❶	❷	❸	❹	❺
$\{a, b, \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ❸	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ❹				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

$\rightarrow$ ❹ 0	{ ❺ 2 }	❺ 2	{ ❷ 3 }
$\rightarrow$ ❺ 0	$\varnothing$	❷ 3	{ ❷ 3 }

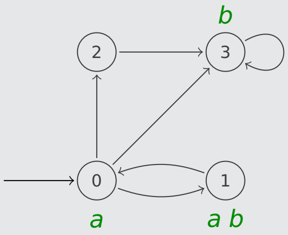
- accepting run ❹ 0 $\xrightarrow{\{a\}}$ ❺ 2 $\xrightarrow{\varnothing}$ ❷ 3 $\xrightarrow{\{b\}}$ ❷ 3 $\xrightarrow{\{b\}}$ $\cdots$

Example

GBA $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	❶	❷	❸	❹	❺
$\{a,\ b,\ \varphi\}$ ❶	✓	✓	✓	✓	✓
$\{\neg a,\ b,\ \varphi\}$ ❷	✓	✓	✓	✓	✓
$\{a,\ \neg b,\ \varphi\}$ ❸	✓	✓	✓		
$\to\ \{a,\ \neg b,\ \neg\varphi\}$ ❹				✓	✓
$\to \{\neg a,\ \neg b,\ \neg\varphi\}$ ❺	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition $\{\{$❶, ❷, ❹, ❺$\}\}$

- product automaton $A_{\neg\varphi} \times A_{\mathcal{M},0}$

$\to$ ❹ 0 | { ❺ 2 }   ❺ 2 | { ❷ 3 }

$\to$ ❺ 0 | $\varnothing$ ❷ 3 | { ❷ 3 }

- accepting run ❹ 0 $\xrightarrow{\{a\}}$ ❺ 2 $\xrightarrow{\varnothing}$ ❷ 3 $\xrightarrow{\{b\}}$ ❷ 3 $\xrightarrow{\{b\}}$ $\cdots$ $\implies$ $\mathcal{M}, 0 \nvDash \varphi$

Outline

1. Summary of Previous Lecture
2. LTL Model Checking
3. Intermezzo
4. LTL Model Checking
5. **Further Reading**

Baier and Katoen

- Section 5.2 of Principles of Model Checking (MIT Press 2008)

Baier and Katoen

- Section 5.2 of Principles of Model Checking (MIT Press 2008)

Important Concepts

- $A_{\mathcal{M},s}$
- A_{φ}
- elementary set

Baier and Katoen

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Important Concepts

- $A_{\mathcal{M},s}$
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- elementary set

homework for January 16

Baier and Katoen

- Section 5.2 of Principles of Model Checking (MIT Press 2008)

Important Concepts

- $A_{\mathcal{M},s}$
- A_{φ}
- elementary set

homework for January 16

next lecture (January 19): online evaluation in presence

Baier and Katoen

- Section 5.2 of Principles of Model Checking (MIT Press 2008)

Important Concepts

- $A_{\mathcal{M},s}$
- A_{φ}
- elementary set

homework for January 16

next lecture (January 19): online evaluation in presence $\implies$ bring device