universität innsbruck

Automata and Logic

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Outline

1. **Summary of Previous Lecture**
2. **Intermezzo**
3. **Recommendations**
4. **Exam**

Definitions

- $\mathbb{B}^+(Q)$ is set of positive boolean formulas over Q:
  - $Q \subseteq \mathbb{B}^+(Q)$
  - $\bot, \top \in \mathbb{B}^+(Q)$
  - if $\varphi, \psi \in \mathbb{B}^+(Q)$ then $\varphi \vee \psi \in \mathbb{B}^+(Q)$
  - if $\varphi, \psi \in \mathbb{B}^+(Q)$ then $\varphi \wedge \psi \in \mathbb{B}^+(Q)$
- alternating finite automaton (AFA) is quintuple $M = (Q, \Sigma, \Delta, s, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\Delta : Q \times \Sigma \to \mathbb{B}^+(Q)$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states

Definitions

- subset $M \subseteq Q$ satisfies formula $\varphi \in \mathbb{B}^+(Q)$ $(M \vDash \varphi)$:

$$M \vDash q \iff q \in M \qquad M \vDash \top \qquad M \nvDash \bot$$

$$M \vDash \varphi \vee \psi \iff M \vDash \varphi \text{ or } M \vDash \psi \qquad M \vDash \varphi \wedge \psi \iff M \vDash \varphi \text{ and } M \vDash \psi$$

- run of AFA $M = (Q, \Sigma, \Delta, s, F)$ on string $a_1 \cdots a_n$ is finite Q–labeled tree r such that
  - all paths from root node to leaf node have length at most n
  - s is label of root node v: $r(v) = s$
  - for every node v on level i $(0 \leqslant i \leqslant n - 1)$ with children $v_1, \ldots, v_k$

$$\{r(v_1), \ldots, r(v_k)\} \vDash \Delta(v, a_{i+1})$$

run is accepting if all leaf nodes are labeled with final states

Theorem

set A of finite strings is regular $\iff$ A is accepted by AFA

Definition

- alternating Büchi automaton (ABA) is quintuple $M = (Q, \Sigma, \Delta, s, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\Delta \colon Q \times \Sigma \to \mathbb{B}^+(Q)$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states
- run of ABA $M = (Q, \Sigma, \Delta, s, F)$ on infinite string $a_1 a_2 a_3 \cdots$ is Q–labeled tree r such that
  - s is label of root node v: $r(v) = s$
  - for every node v on level i with children $v_1, \dots, v_k$

$$\{r(v_1), \dots, r(v_k)\} \vDash \Delta(v, a_{i+1})$$

  run is accepting if all infinite paths π of r are accepting $(\infty(\pi) \cap F \neq \varnothing)$
- $L(M) = \{x \in \Sigma^\omega \mid x \text{ has accepting run}\}$

Theorem

ABAs are effectively closed under complement, intersection and union

Theorem

every ABA can be effectively transformed into equivalent NBA

Lemma

every LTL formula can be transformed into equivalent negation normal form (NNF):

$$\varphi \ ::= \bot \mid \top \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{X}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \varphi\,\mathsf{R}\,\varphi$$

LTL formula φ in negation normal form with atoms in AP

Definitions

- (positive) closure $\mathcal{C}_+(\varphi)$ of φ consists of all subformulas of φ
- ABA $A_\varphi = (\mathcal{C}_+(Q), 2^{\mathsf{AP}}, \Delta, \varphi, F)$ with $F = \{\psi \in Q \mid \psi = \varphi_1 \,\mathsf{R}\, \varphi_2\}$ and

$$\Delta(p, A) = \begin{cases} \top & \text{if } p \in A \\ \bot & \text{if } p \notin A \end{cases} \qquad \Delta(\neg p, A) = \begin{cases} \top & \text{if } p \notin A \\ \bot & \text{if } p \in A \end{cases}$$

$$\Delta(\varphi_1 \wedge \varphi_2, A) = \Delta(\varphi_1, A) \wedge \Delta(\varphi_2, A) \qquad \Delta(\top, A) = \top$$

$$\Delta(\varphi_1 \vee \varphi_2, A) = \Delta(\varphi_1, A) \vee \Delta(\varphi_2, A) \qquad \Delta(\bot, A) = \bot$$

$$\Delta(\mathsf{X}\,\psi, A) = \psi$$

$$\Delta(\varphi_1 \,\mathsf{U}\, \varphi_2, A) = \Delta(\varphi_2, A) \vee (\Delta(\varphi_1, A) \wedge \varphi_1 \,\mathsf{U}\, \varphi_2)$$

$$\Delta(\varphi_1 \,\mathsf{R}\, \varphi_2, A) = \Delta(\varphi_2, A) \wedge (\Delta(\varphi_1, A) \vee \varphi_1 \,\mathsf{R}\, \varphi_2)$$

Theorem

$L(A_\varphi) = L(\varphi)$

Remark

not every ω–regular subset of $(2^{\mathsf{AP}})^\omega$ is expressible in LTL

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

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Question

Which of the following statements are true ?

A WMSO definable sets are ω–regular.

B The emptiness problem of GBAs is undecidable.

C The set $\{p, \neg\,\mathrm{X}\,q, \neg p\,\mathrm{U}\,(\mathrm{X}\,q)\} \subseteq \mathcal{C}(\neg p\,\mathrm{U}\,(\mathrm{X}\,q))$ is elementary.

D The WMSO formula $\exists X.\forall x.X(x) \to \exists y.x < y \wedge X(y)$ is valid.

E The LTL formulas $(\mathrm{G}\,p)\,\mathrm{R}\,q$ and $q\,\mathrm{W}\,(q \wedge \neg(\top\,\mathrm{U}\,\neg p))$ are equivalent.

F The formula $\exists y.x \times x = y + y$ is expressible in Presburger arithmetic.

G Given an alphabet Σ, the set $\{xy \mid x \neq y \text{ and } x, y \in \Sigma^*\}$ is WSMO definable.

evaluation

homework for January 30

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Recommended Courses in 26S

- WM 2: Constraint Solving
- WM 7: Interactive Theorem Proving in Isabelle / HOL
- WM 7: Current Challenges in Probabilistic Learning, Inference, and their Applications
- WM 8: Advanced Logic and Quantum Logic
- WM 9: Research Seminar CL / TCS
- WM 20: Term Rewriting

Master Projects

`http://cl-informatik.uibk.ac.at/teaching/master/available.php`

Outline

First Exam on February 2

- closed book
- second exam on February 26, third exam on September 25 (on demand)

Preparation

- review exams and solutions of 22W / 23W / 24W courses

exam 1 (22W)	solutions	exam 2 (22W)	solutions
exam 1 (23W)	solutions	exam 2 (23W)	solutions
exam 1 (24W)	solutions	exam 2 (24W)	solutions

- review homework exercises and solutions, in particular

lecture 4	exercise 1	lecture 7	exercise 1, 3	lecture 9b	exercise 2, 3
lecture 5	exercise 1, 2, 3, 4	lecture 8	exercise 2	lecture 12	exercise 2
lecture 6	exercise 1	lecture 9a	exercise 5	lecture 13	exercise 4

- study slides (and visit consultation hours)