

Automata and Logic

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Outline

1. Summary of Previous Lecture
2. Intermezzo
3. Recommendations
4. Exam

Definitions

- ▶ $\mathbb{B}^+(Q)$ is set of positive boolean formulas over Q :
 - ▶ $Q \subseteq \mathbb{B}^+(Q)$
 - ▶ $\perp, \top \in \mathbb{B}^+(Q)$
 - ▶ if $\varphi, \psi \in \mathbb{B}^+(Q)$ then $\varphi \vee \psi \in \mathbb{B}^+(Q)$
 - ▶ if $\varphi, \psi \in \mathbb{B}^+(Q)$ then $\varphi \wedge \psi \in \mathbb{B}^+(Q)$
- ▶ **alternating finite automaton (AFA)** is quintuple $M = (Q, \Sigma, \Delta, s, F)$ with
 - ① Q : finite set of states
 - ② Σ : input alphabet
 - ③ $\Delta: Q \times \Sigma \rightarrow \mathbb{B}^+(Q)$: transition function
 - ④ $s \in Q$: start state
 - ⑤ $F \subseteq Q$: final (accept) states

Definitions

- ▶ subset $M \subseteq Q$ **satisfies** formula $\varphi \in \mathbb{B}^+(Q)$ ($M \models \varphi$):

$$M \models q \iff q \in M \qquad M \models \top \qquad M \not\models \perp$$

$$M \models \varphi \vee \psi \iff M \models \varphi \text{ or } M \models \psi \qquad M \models \varphi \wedge \psi \iff M \models \varphi \text{ and } M \models \psi$$
- ▶ **run** of AFA $M = (Q, \Sigma, \Delta, s, F)$ on string $a_1 \cdots a_n$ is finite Q -labeled tree r such that
 - ▶ all paths from root node to leaf node have length at most n
 - ▶ s is label of root node v : $r(v) = s$
 - ▶ for every node v on level i ($0 \leq i \leq n-1$) with children $v_1, \dots, v_k$

$$\{r(v_1), \dots, r(v_k)\} \models \Delta(v, a_{i+1})$$
- run is **accepting** if all leaf nodes are labeled with final states

Theorem

set A of finite strings is regular $\iff A$ is accepted by AFA

Definition

- ▶ **alternating Büchi automaton (ABA)** is quintuple $M = (Q, \Sigma, \Delta, s, F)$ with
 - ① Q : finite set of states
 - ② Σ : input alphabet
 - ③ $\Delta: Q \times \Sigma \rightarrow \mathbb{B}^+(Q)$: transition function
 - ④ $s \in Q$: start state
 - ⑤ $F \subseteq Q$: final (accept) states
- ▶ **run** of ABA $M = (Q, \Sigma, \Delta, s, F)$ on infinite string $a_1 a_2 a_3 \dots$ is Q -labeled tree r such that
 - ▶ s is label of root node v : $r(v) = s$
 - ▶ for every node v on level i with children $v_1, \dots, v_k$

$$\{r(v_1), \dots, r(v_k)\} \models \Delta(v, a_{i+1})$$
- ▶ run is **accepting** if all infinite paths π of r are accepting ($\infty(\pi) \cap F \neq \emptyset$)
- ▶ $L(M) = \{x \in \Sigma^\omega \mid x \text{ has accepting run}\}$

Theorem

ABAs are effectively closed under complement, intersection and union

Theorem

every ABA can be effectively transformed into equivalent NBA

Lemma

every LTL formula can be transformed into equivalent **negation normal form (NNF)**:

$$\varphi ::= \perp \mid \top \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid X\varphi \mid \varphi U \varphi \mid \varphi R \varphi$$

LTL formula φ in negation normal form with atoms in AP

Definitions

- ▶ **(positive) closure $\mathcal{C}_+(\varphi)$** of φ consists of all subformulas of φ
- ▶ ABA $A_\varphi = (\mathcal{C}_+(Q), 2^{AP}, \Delta, \varphi, F)$ with $F = \{\psi \in Q \mid \psi = \varphi_1 R \varphi_2\}$ and

$$\Delta(p, A) = \begin{cases} \top & \text{if } p \in A \\ \perp & \text{if } p \notin A \end{cases} \quad \Delta(\neg p, A) = \begin{cases} \top & \text{if } p \notin A \\ \perp & \text{if } p \in A \end{cases}$$

$$\Delta(\varphi_1 \wedge \varphi_2, A) = \Delta(\varphi_1, A) \wedge \Delta(\varphi_2, A) \quad \Delta(\top, A) = \top$$

$$\Delta(\varphi_1 \vee \varphi_2, A) = \Delta(\varphi_1, A) \vee \Delta(\varphi_2, A) \quad \Delta(\perp, A) = \perp$$

$$\Delta(X\psi, A) = \psi$$

$$\Delta(\varphi_1 U \varphi_2, A) = \Delta(\varphi_2, A) \vee (\Delta(\varphi_1, A) \wedge \varphi_1 U \varphi_2)$$

$$\Delta(\varphi_1 R \varphi_2, A) = \Delta(\varphi_2, A) \wedge (\Delta(\varphi_1, A) \vee \varphi_1 R \varphi_2)$$

Theorem

$$L(A_\varphi) = L(\varphi)$$

Remark

not every ω -regular subset of $(2^{AP})^\omega$ is expressible in LTL

Automata

- ▶ (deterministic, nondeterministic, alternating) finite automata
- ▶ regular expressions
- ▶ (alternating) Büchi automata

Logic

- ▶ (weak) monadic second-order logic
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

 with session ID **4957 9500**

Question

Which of the following statements are true ?

- A** WMSO definable sets are ω -regular.
- B** The emptiness problem of GBAs is undecidable.
- C** The set $\{p, \neg Xq, \neg p \cup (Xq)\} \subseteq \mathcal{C}(\neg p \cup (Xq))$ is elementary.
- D** The WMSO formula $\exists X. \forall x. X(x) \rightarrow \exists y. x < y \wedge X(y)$ is valid.
- E** The LTL formulas $(Gp)Rq$ and $qW(q \wedge \neg(\top U \neg p))$ are equivalent.
- F** The formula $\exists y. x \times x = y + y$ is expressible in Presburger arithmetic.
- G** Given an alphabet Σ , the set $\{xy \mid x \neq y \text{ and } x, y \in \Sigma^*\}$ is WMSO definable.



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evaluation

homework for January 30

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Recommended Courses in 26S

Logic and Learning

- ▶ WM 2: Constraint Solving
- ▶ WM 7: Interactive Theorem Proving in Isabelle/HOL
- ▶ WM 7: Current Challenges in Probabilistic Learning, Inference, and their Applications
- ▶ WM 8: Advanced Logic and Quantum Logic
- ▶ WM 9: Research Seminar CL/TCS
- ▶ WM 20: Term Rewriting

Master Projects

<http://cl-informatik.uibk.ac.at/teaching/master/available.php>

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First Exam on February 2

- ▶ closed book
- ▶ second exam on February 26, third exam on September 25 (on demand)

Preparation

- ▶ review exams and solutions of 22W/23W/24W courses
 - exam 1 (22W) solutions exam 2 (22W) solutions
 - exam 1 (23W) solutions exam 2 (23W) solutions
 - exam 1 (24W) solutions exam 2 (24W) solutions
- ▶ review homework exercises and solutions, in particular
 - lecture 4 exercise 1 lecture 7 exercise 1, 3 lecture 9b exercise 2, 3
 - lecture 5 exercise 1, 2, 3, 4 lecture 8 exercise 2 lecture 12 exercise 2
 - lecture 6 exercise 1 lecture 9a exercise 5 lecture 13 exercise 4
- ▶ study slides (and visit consultation hours)