Selected Solutions

$\boxed{1}$ Consider the following interpretations in $\mathbb{N} \setminus \{0,1\}$: $\mathsf{f}_{\mathbb{N}}(x,y) = xy$ and $\mathsf{g}_{\mathbb{N}}(x,y) = 2x + y + 1$. Since

$$\begin{aligned} x(2y+z+1) &> 2xy + xz + 1 \\ (2x+y+1)z &> 2xz + yz + 1 \\ 2(2x+y+1)+z+1 &> 2x + (2y+z+1) + 1 \end{aligned}$$

for all $x, y, z \geqslant 2$, the TRS is polynomially terminating. If we change the interpretations of f and g to $\mathsf{f}_{\mathbb{N}}(x,y) = xy + 2x + 2y + 2$ and $\mathsf{g}_{\mathbb{N}}(x,y) = 2x + y + 5$ then the resulting inequalities are satisfied for all $x, y, z \in \mathbb{N}$.

$\boxed{2}$ Consider the well-founded monotone algebra $(\mathcal{A}, >_{\mathbb{N}})$ with carrier $\mathbb{N} \setminus \{0\}$ and interpretations $\mathsf{f}_{\mathcal{A}}(x,y) = 3^x + y$ and $\mathsf{g}_{\mathcal{A}}(x) = x + 1$. Since

$$\begin{aligned} 3^{x+1} + y &>_{\mathbb{N}} 3^x + (3^x + y) + 1 \\ 3^x + x &>_{\mathbb{N}} x + 2 \end{aligned}$$

for all $x, y \in \mathbb{N} \setminus \{0\}$, the TRS is terminating.

$\boxed{3}\boxed{\text{(b)}}$ Consider the TRS $\mathcal{R}_k$ consisting of the two rewrite rules

$$\mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad\qquad \mathsf{a} \to \mathsf{g}^k(\mathsf{b})$$

for $k \geqslant 0$. By taking the weight function

$$w(\mathsf{f}) = 0 \qquad\qquad w(\mathsf{g}) = w(\mathsf{b}) = w_0 = 1 \qquad\qquad w(\mathsf{a}) = k + 1$$

together with the admissible precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{b}$ we obtain $\mathcal{R} \subseteq >_{\mathsf{kbo}}$. In the following we argue that the weight of a must be at least $k+1$. The two sides of the first rewrite rule have the same weight, regardless of the weight function, and hence $\mathsf{f} > \mathsf{g}$. Since g is unary, it cannot have weight zero as this would contradict admissibility. We have $w(\mathsf{b}) > 0$ by definition of weight function. It follows that the weight of the term $\mathsf{g}^k(\mathsf{b})$ is at least $k + 1$. To orient the second rule with KBO, we thus must have $w(\mathsf{a}) \geqslant k + 1$.