Selected Solutions

$\boxed{2}$ $\boxed{\text{(a)}}$ Let $p = \deg(P)$ and $q = \deg(Q)$. The result is trivial if $p = 0$, since then P is a constant and hence $P = Q$. Let $p > 0$. From $Q(x) = P(x, \ldots, x)$ it follows that P contains at least one monomial of degree q, implying $p \geqslant q$. We can split P into the polynomials $P_1(x_1, \ldots, x_n)$ which contains all monomials of P of degree p and $P_2(x_1, \ldots, x_n)$ consisting of the remaining monomials of degree strictly less than p. Therefore, $P(x_1, \ldots, x_n) = P_1(x_1, \ldots, x_n) + P_2(x_1, \ldots, x_n)$, where $P_1 \neq 0$ since $p > 0$. So there are $b_1, \ldots, b_n \geqslant 0$ with $P_1(b_1, \ldots, b_n) \neq 0$. Define $c = \max\{b_1, \ldots, b_n\}$. We conclude $Q(ca) = P(ca, \ldots, ca) \geqslant P(b_1 a, \ldots, b_n a) \geqslant P(0, \ldots, 0)$ for every $a \geqslant 0$ by using monotonicity of P. Therefore,

$$\begin{aligned} q \;&=\; \deg(Q(x)) \;=\; \deg(Q(cx)) \;\geqslant\; \deg(P(b_1 x, \ldots, b_n x)) \\ &=\; \deg(P_1(b_1 x, \ldots, b_n x) + P_2(b_1 x, \ldots, b_n x)) \\ &=\; \deg(P_1(b_1, \ldots, b_n) \cdot x^p + P_2(b_1 x, \ldots, b_n x)) \;=\; p \end{aligned}$$

where the last equality is a consequence of $P_1(b_1, \ldots, b_n) \neq 0$ and $\deg(P_2(b_1 x, \ldots, b_n x)) < p$. Since $q \leqslant p$ and $q \geqslant p$ we arrive at $\deg(Q) = q = p = \deg(P)$.

$\boxed{3}$ Consider the following strictly monotone interpretation in $\mathbb{N}$: $\mathsf{a}_{\mathbb{N}}(x) = 3^x$, $\mathsf{b}_{\mathbb{N}}(x) = 2x$ and $\mathsf{c}_{\mathbb{N}}(x) = x + 1$. Since

$$\mathsf{a}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(x))) \;=\; 3^{2x+2} \;>\; 2 \cdot 3^{2x+1} \;=\; \mathsf{b}_{\mathbb{N}}(\mathsf{a}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}}(x))))$$

for all $x \geqslant 0$, the SRS is ω-terminating. Assume the SRS is polynomially terminating. So there exist strictly monotone univariate polynomials $\mathsf{a}_{\mathbb{N}}$, $\mathsf{b}_{\mathbb{N}}$ and $\mathsf{c}_{\mathbb{N}}$ such that $\mathsf{a}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(x))) > \mathsf{b}_{\mathbb{N}}(\mathsf{a}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}}(x))))$ for all $x \geqslant 0$. In particular, $\deg(\mathsf{a}_{\mathbb{N}} \circ \mathsf{b}_{\mathbb{N}} \circ \mathsf{c}_{\mathbb{N}}) \geqslant \deg(\mathsf{b}_{\mathbb{N}} \circ \mathsf{a}_{\mathbb{N}} \circ \mathsf{c}_{\mathbb{N}} \circ \mathsf{b}_{\mathbb{N}})$ which gives $\deg(\mathsf{b}_{\mathbb{N}}) = 1$. So $\mathsf{b}_{\mathbb{N}}(x) = ax + b$ for some $a \geqslant 1$ and $b > 0$. Moreover, the leading coefficient of $\mathsf{a}_{\mathbb{N}} \circ \mathsf{b}_{\mathbb{N}} \circ \mathsf{c}_{\mathbb{N}}$ cannot be smaller than that of $\mathsf{b}_{\mathbb{N}} \circ \mathsf{a}_{\mathbb{N}} \circ \mathsf{c}_{\mathbb{N}} \circ \mathsf{b}_{\mathbb{N}}$ and thus $a = 1$. Using strict monotonicity one easily proves by induction on n that $\mathsf{b}_{\mathbb{N}}(n) \geqslant n$ and $\mathsf{c}_{\mathbb{N}}(m + n) \geqslant \mathsf{c}_{\mathbb{N}}(m) + n$ for all $m, n \in \mathbb{N}$. Hence

$$\mathsf{b}_{\mathbb{N}}(\mathsf{a}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}}(x)))) \;\geqslant\; \mathsf{a}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}}(x))) \;=\; \mathsf{a}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(x + b)) \;\geqslant\; \mathsf{a}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(x) + b) \;=\; \mathsf{a}_{\mathbb{N}}(\mathsf{b}_{\mathbb{N}}(\mathsf{c}_{\mathbb{N}}(x)))$$

contradicting the assumption. So the given SRS is not polynomially terminating.