1 (a) No, bacause $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is non-terminating:

$$\mathsf{g}(\mathsf{b}) \to_{\mathcal{R}} \mathsf{g}(\mathsf{f}(\mathsf{a})) \to_{\mathcal{R}} \mathsf{g}(\mathsf{f}(\mathsf{b})) \to_{\mathcal{E}\mathsf{mb}} \mathsf{g}(\mathsf{b})$$

(b) Note that both rules of $\mathcal{R}$ are root-preserving. Hence $\mathcal{R}$ is terminating if and only if $\mathcal{R}_{\mathsf{rl}}$ is terminating. The latter TRS consists of the rules

$$\mathsf{f}_{\mathsf{a}}(\mathsf{a}) \to \mathsf{f}_{\mathsf{b}}(\mathsf{b}) \qquad \mathsf{g}_{\mathsf{b}}(\mathsf{b}) \to \mathsf{g}_{\mathsf{f}}(\mathsf{f}_{\mathsf{a}}(\mathsf{a}))$$

and is compatible with LPO for the precedence $\mathsf{g}_{\mathsf{b}} > \mathsf{g}_{\mathsf{f}} > \mathsf{f}_{\mathsf{a}} > \mathsf{f}_{\mathsf{b}} > \mathsf{a} > \mathsf{b}$.

(c) The TRS $\mathcal{R}_1 = \mathcal{R} \cup \{\mathsf{f}(x) \to \mathsf{g}(\mathsf{g}(x))\}$ is not terminating:

$$\underline{\mathsf{g}(\mathsf{b})} \to \mathsf{g}(\mathsf{f}(\mathsf{a})) \to \mathsf{g}(\mathsf{f}(\mathsf{b})) \to \mathsf{g}(\mathsf{g}(\underline{\mathsf{g}(\mathsf{b})}))$$

The TRS $\mathcal{R}_2 = \mathcal{R} \cup \{\mathsf{g}(x) \to \mathsf{f}(\mathsf{f}(x))\}$ is terminating. We show this using semantic labeling. Consider the algebra $\mathcal{A}$ with carrier $A = \{0,1\}$ and interpretations $\mathsf{f}_{\mathcal{A}}(x) = \mathsf{g}_{\mathcal{A}}(x) = \mathsf{a}_{\mathcal{A}} = 0$ and $\mathsf{b}_{\mathcal{A}} = 1$. Clearly $\mathcal{A}$ is a model for $\mathcal{R}_2$. Define the labeling (L, lab) for $\mathcal{A}$ as follows: $L_{\mathsf{f}} = L_{\mathsf{g}} = A$ and

$$\mathsf{lab}_{\mathsf{f}}(x) = \mathsf{lab}_{\mathsf{g}}(x) = x$$

The TRS $\mathcal{R}_{\mathsf{lab}}$

$$\mathsf{f}_0(\mathsf{a}) \to \mathsf{f}_1(\mathsf{b}) \qquad \mathsf{g}_1(\mathsf{b}) \to \mathsf{g}_0(\mathsf{f}_0(\mathsf{a})) \qquad \mathsf{g}_0(x) \to \mathsf{f}_0(\mathsf{f}_0(x)) \qquad \mathsf{g}_1(x) \to \mathsf{f}_0(\mathsf{f}_1(x))$$

is compatible with LPO for the precedence $\mathsf{g}_1 > \mathsf{g}_0 > \mathsf{f}_0 > \mathsf{f}_1 > \mathsf{b} > \mathsf{a}$. Hence $\mathcal{R}_2$ is terminating. Root-labeling can also be used to prove the terminating of $\mathcal{R}_2$.

2 (a) The TRS $\mathcal{R}$ has six extended rules:

$$((x \oplus y) \cdot z) \cdot w \to (x \cdot z \oplus y \cdot z) \cdot w \qquad (x \cdot x) \cdot w \to x \cdot w$$
$$(1 \cdot x) \cdot w \to x \cdot w \qquad (\mathsf{0} \oplus x) \oplus w \to x \oplus w$$
$$(x \oplus x) \oplus w \to \mathsf{0} \oplus w \qquad (\mathsf{0} \cdot x) \cdot w \to \mathsf{0} \cdot w$$

(b) We have

$$\begin{aligned} t &\to_{\mathcal{R}} (x \cdot y) + \overline{y \oplus x} \\ &\to_{\mathcal{R}} (x \cdot y) + ((y \oplus x) \oplus 1) \\ &\to_{\mathcal{R}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus ((x \cdot y) \cdot ((y \oplus x) \oplus 1)) \in \mathsf{NF}(\mathcal{R}) \end{aligned}$$

and

$$\begin{aligned} t &\to_{\mathcal{R}^e,\mathsf{AC}} (x \cdot y) + \overline{y \oplus x} \\ &\to_{\mathcal{R}^e,\mathsf{AC}} (x \cdot y) + ((y \oplus x) \oplus 1) \\ &\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus ((x \cdot y) \cdot ((y \oplus x) \oplus 1)) \\ &\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus (((y \oplus x) \cdot (x \cdot y)) \oplus (1 \cdot (x \cdot y))) \end{aligned}$$

$$
\begin{aligned}
&\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus (((y \oplus x) \cdot (x \cdot y)) \oplus (x \cdot y))\\
&\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus (((y \cdot (x \cdot y)) \oplus (x \cdot (x \cdot y))) \oplus (x \cdot y))\\
&\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus (((y \cdot x) \oplus (x \cdot (x \cdot y))) \oplus (x \cdot y))\\
&\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus (((y \cdot x) \oplus (x \cdot y)) \oplus (x \cdot y))\\
&\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus (\mathsf{0} \oplus (x \cdot y))\\
&\to_{\mathcal{R}^e,\mathsf{AC}} ((x \cdot y) \oplus ((y \oplus x) \oplus 1)) \oplus (x \cdot y)\\
&\to_{\mathcal{R}^e,\mathsf{AC}} \mathsf{0} \oplus ((y \oplus x) \oplus 1)\\
&\to_{\mathcal{R}^e,\mathsf{AC}} (y \oplus x) \oplus 1 \in \mathsf{NF}(\mathcal{R}^e, \mathsf{AC})
\end{aligned}
$$

(c) Consider the following polynomial interpretation:

$$
\begin{aligned}
\mathsf{0}_{\mathbb{N}} = \mathsf{1}_{\mathbb{N}} &= 0 & \cdot_{\mathbb{N}}(x,y) &= 2xy + 2x + 2y + 1\\
\overline{x}^{\mathbb{N}} &= x + 3 & +_{\mathbb{N}}(x,y) &= 2xy + 3x + 3y + 6\\
\oplus_{\mathbb{N}}(x,y) &= x + y + 2 & \Rightarrow_{\mathbb{N}}(x,y) = \Leftrightarrow_{\mathbb{N}}(x,y) &= x + y + 6
\end{aligned}
$$

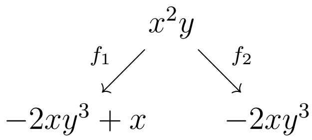
Since

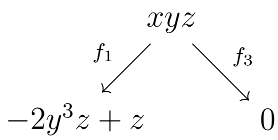
$$
\begin{aligned}
2x + 1 &> 0 & x + 3 &> x + 2\\
2x + 1 &> x & 2xy + 3x + 3y + 6 &> 2xy + 3x + 3y + 5\\
2x^2 + 4x + 1 &> x & x + y + 6 &> x + y + 5\\
x + 2 &> x & x + y + 6 &> x + y + 5\\
2x + 2 &> 0 & 2xz + 2yz + 2x + 2y + 6z + 5 &> 2xz + 2yz + 2x + 2y + 4z + 4
\end{aligned}
$$

for all $x, y, z \geqslant 0$ and both $x + y + 2$ and $2xy + 2x + 2y + 1$ are AC compatible, the TRS is AC terminating.

(d) No. The rewrite rule $x + y \to (x \oplus y) \oplus (x \cdot y)$ demands $+ > \oplus$, but then the AC symbol $+$ cannot be minimal in the precedence.

3 (a) The PRS $(\{f_1, f_2, f_3\}, >^{\mathsf{l}})$ with $f_1 = xy + 2y^3 - 1$, $f_2 = x^2 + 2xy^2$ and $f_3 = yz$ has two critical pairs:





(b) We use the inference system GB. The following run is non-failing and simplifying:

1		~~$xy + 2y^3 - 1$~~ [4]	
2		~~$x^2 + 2xy^2$~~ [5]	
3		~~yz~~ [6]	
4	$\mathsf{orient}(1)$		~~$xy \to -2y^3 + 1$~~ [13]
5	$\mathsf{simplify}_3(2,4)$	~~$x^2 - 4y^4 + 2y$~~ [11]	
6	$\mathsf{orient}(3)$		~~$yz \to 0$~~ [10]
7	$\mathsf{deduce}(4,6)$	~~$-2y^3z + z$~~ [8]	
8	$\mathsf{simplify}_3(7,6)$	~~z~~ [9]	
9	$\mathsf{orient}(8)$		$z \to 0$
10	$\mathsf{simplify}_2(6,9)$		
11	$\mathsf{orient}(5)$		~~$x^2 \to 4y^4 - 2y$~~ [17]

12	$\mathsf{deduce}(4, 11)$	~~$-2xy^3 + x - 4y^5 + 2y^2$~~ [13]	
13	$\mathsf{simplify}_3(12, 4)$	~~x~~ [14]	
14	$\mathsf{orient}(14)$		$x \to 0$
15	$\mathsf{simplify}_2(4, 14)$		~~$2y^3 \to 1$~~ [16]
16	$\mathsf{simplify}_1(15)$		$y^3 \to \frac{1}{2}$
17	$\mathsf{simplify}_2(11, 14)$		~~$-4y^4 \to -2y$~~ [18]
18	$\mathsf{simplify}_2(17, 16)$		

The final PRS $G = (\{x, y^3 - \frac{1}{2}, z\}, >^{\mathsf{l}})$ is a canonical Gröbner basis equivalent to F.

(c) We have

$$xz + y^3 \to_G^! \tfrac{1}{2} \neq 0 \qquad\qquad 2x^2y^2 - z^3 \to_G^! 0$$

and hence $xz + y^3 \notin \text{ideal}(F)$ and $2x^2y^2 - z^3 \in \text{ideal}(F)$.

4 Statements 3, 5, 6, 9 and 10 are true, the other statements are false.

1. Every PRS is confluent.

This is false. A PRS is confluent if and only if it is Gröbner basis.

2. $(\omega^5 + \omega) \cdot (\omega^3 + 2) = \omega^8 + \omega^5 + \omega$

This is false. We have $(\omega^5 + \omega) \cdot (\omega^3 + 2) = \omega^8 + \omega^5 \cdot 2 + \omega$ instead.

3. Every partial well-order is well-founded.

This is true.

4. Linear termination is decidable for one-rule TRSs.

This is false.

5. Every precedence terminating TRS is simply terminating.

This is true. Precedence terminating TRSs are compatible with $>_{\mathsf{lpo}}$ for a well-founded precedence $>$ and thus simply terminating. (This also holds for infinite signatures due to the incrementality of LPO.)

6. Every admissible order on power products is well-founded.

This is true. Well-foundedness is part of the definition.

7. Ordinal multiplication is strictly monotone in its first argument.

This is false. We have $2 > 1$ and $2 \cdot \omega = \omega = 1 \cdot \omega$. Weak monotonicity does hold.

8. Every totally terminating TRS over a finite signature is ω-terminating.

This is false. The reverse direction holds.

9. A TRS is terminating if and only if its flat context closure is terminating.

This is true.

10. Every finite simply terminating TRS is compatible with a suitable weighted path order.

This is true.