This test consists of four exercises. The available points for each item are written in the margin. ***Explain your answers!***

1 Consider the TRS $\mathcal{R}$ consisting of the two rewrite rules

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad\qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{f}(\mathsf{a}))$$

⟨5⟩ (a) Is $\mathcal{R}$ simply terminating?

⟨5⟩ (b) Use root-labeling to prove that $\mathcal{R}$ is terminating.

⟨10⟩ (c) Exactly one of the TRSs

$$\mathcal{R} \cup \{\mathsf{f}(x) \to \mathsf{g}(\mathsf{g}(x))\} \qquad\qquad \mathcal{R} \cup \{\mathsf{g}(x) \to \mathsf{f}(\mathsf{f}(x))\}$$

is terminating. Which one? Prove your answer.

2 Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$\begin{aligned} \mathsf{0} \cdot x &\to \mathsf{0} & \overline{x} &\to x \oplus \mathsf{1} \\ \mathsf{1} \cdot x &\to x & x + y &\to (x \oplus y) \oplus (x \cdot y) \\ x \cdot x &\to x & x \Rightarrow y &\to \overline{x} + y \\ \mathsf{0} \oplus x &\to x & x \Leftrightarrow y &\to \overline{x \oplus y} \\ x \oplus x &\to \mathsf{0} & (x \oplus y) \cdot z &\to (x \cdot z) \oplus (y \cdot z) \end{aligned}$$

with AC symbols $\cdot$ and $\oplus$.

⟨5⟩ (a) Compute all extensions of rules of $\mathcal{R}$.

⟨10⟩ (b) Rewrite the term $t = (x \cdot y) + (y \Leftrightarrow x)$ to normal form with $\to_{\mathcal{R}}$ and with $\to_{\mathcal{R}^e,\mathsf{AC}}$.

⟨10⟩ (c) Prove the AC termination of $\mathcal{R}$ by constructing a suitable polynomial interpretation.

⟨10⟩ (d) Can the AC termination of $\mathcal{R}$ be shown by AC-MPO?

3 Consider the set $F = \{xy + 2y^3 - 1, x^2 + 2xy^2, yz\}$ of polynomials in $\mathbb{Q}[x, y, z]$.

⟨5⟩ (a) Compute all critical pairs of $(F, >^{\mathsf{l}})$.

⟨10⟩ (b) Transform F into an equivalent canonical Gröbner basis.

⟨10⟩ (c) Which of the following polynomials belong to the ideal generated by F?

i. $xz + y^3$

ii. $2x^2y^2 - z^3$

⟨20⟩ [4] Determine whether the following statements are true or false. (Providing explanation is optional.) Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

1. Every PRS is confluent.
2. $(\omega^5 + \omega) \cdot (\omega^3 + 2) = \omega^8 + \omega^5 + \omega$
3. Every partial well-order is well-founded.
4. Linear termination is decidable for one-rule TRSs.
5. Every precedence terminating TRS is simply terminating.
6. Every admissible order on power products is well-founded.
7. Ordinal multiplication is strictly monotone in its first argument.
8. Every totally terminating TRS over a finite signature is ω-terminating.
9. A TRS is terminating if and only if its flat context closure is terminating.
10. Every finite simply terminating TRS is compatible with a suitable weighted path order.