Advanced Topics in Termination

ISR 2009 – lecture 1

Aart Middeldorp

Institute of Computer Science
University of Innsbruck

Example (Addition on Natural Numbers)

signature	$0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)
terms	$1+3 \qquad 2+(7:3) \qquad (2:(3:x))+((1+7):2)$
rewrite rules	see below
rewriting	$(2:3)+(7:7) \to^* (1:0):0$

$$
\begin{array}{llllll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 \\
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

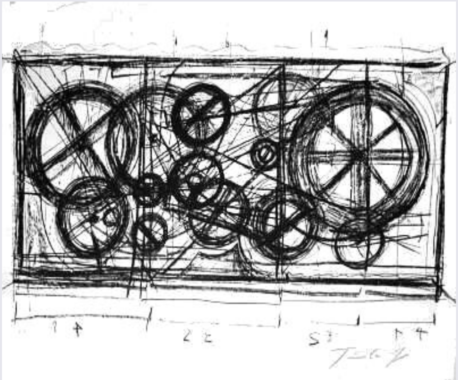
Definition

rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, ...

Termination Research



Termination Tools

CiME, T$_{\text{T}}$T, AProVE, Termptation, Cariboo, Matchbox, Torpa, Teparla, Jambox, TPA, MultumNonMulta, MuTerm, T$_{\text{T}}$Tbox, NTI, T$_{\text{T}}$T$_2$, Nonloop, TrafO, VMTL, ...

Topics

- polynomial interpretations
- matrix interpretations
- dependency pairs
- match-bounds
- semantic labeling
- derivational complexity
- decidable classes
- certification
- applications

Further Reading

- `http://cl-informatik.uibk.ac.at/~ami/09isr/`

Outline

Definitions

- rewrite order is proper order $>$ on terms which is
  - closed under contexts $s > t \implies C[s] > C[t]$
  - closed under substitutions $s > t \implies s\sigma > t\sigma$
- reduction order is well-founded rewrite order

Notation

$\mathcal{R} \subseteq >$ abbreviates $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >$ *for reduction order* $>$

Definitions

- well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of nonempty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) > f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$

- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemma

$>_{\mathcal{A}}$ is reduction order for every well-founded monotone algebra $(\mathcal{A}, >)$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded monotone algebra $(\mathcal{A}, >)$

Well-Founded Monotone Algebras

used in termination proofs/tools:

- polynomial interpretations over $\mathbb{N}$
- polynomial interpretations over $\mathbb{Q}$ and $\mathbb{R}$
- matrix interpretations over $\mathbb{N}$
- matrix interpretations over $\mathbb{N} \cup \{-\infty\}$
- $\dots$

Outline

Example

- rewrite system

$$
\begin{aligned}
0 + y &\to y \\
\mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\
0 \times y &\to 0 \\
\mathsf{s}(x) \times y &\to y + (x \times y)
\end{aligned}
$$

- interpretations

$$
\begin{aligned}
0_{\mathcal{A}} &= 1 \\
\mathsf{s}_{\mathcal{A}}(x) &= x + 1 \\
+_{\mathcal{A}}(x, y) &= 2x + y \\
\times_{\mathcal{A}}(x, y) &= 2xy + x + y + 1
\end{aligned}
$$

- constraints $\quad \forall x, y \in \mathbb{N}$

$$
\begin{aligned}
y + 2 &> y \\
2x + y + 2 &> 2x + y + 1 \\
3y + 2 &> 1 \\
2xy + x + 3y + 2 &> 2xy + x + 3y + 1
\end{aligned}
$$

Definition

TRS $\mathcal{R}$ is polynomially terminating over $\mathbb{N}$ if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of $\mathcal{A}$ is $\mathbb{N}$
- $>$ is standard order on $\mathbb{N}$
- $f_{\mathcal{A}} \in \mathbb{Z}[x_1, \ldots, x_n]$ for every n-ary f

polynomials with coefficients in $\mathbb{Z}$ and indeterminates $x_1, \ldots, x_n$

Lemma

$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$

$\iff$

$\mathcal{R}$ is polynomially terminating over $\{n \in \mathbb{N} \mid n \geqslant N\}$ for some $N \geqslant 0$

Questions

1. how to find suitable polynomials ?
2. how to show that $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$?

Theorem

following problem is undecidable:

instance: polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$

question: $\forall x_1, \ldots, x_n \in \mathbb{N}\colon P(x_1, \ldots, x_n) > 0$?

Proof

reduction from Hilbert's 10th Problem

for arbitrary polynomial $Q \in \mathbb{Z}[x_1, \ldots, x_n]$

$\exists x_1, \ldots, x_n \in \mathbb{Z}\colon Q(x_1, \ldots, x_n) = 0$

$\iff \quad \neg\forall x_1, \ldots, x_n \in \mathbb{Z}\colon Q(x_1, \ldots, x_n) \neq 0$

$\iff \quad \neg\forall x_1, \ldots, x_n \in \mathbb{Z}\colon Q(x_1, \ldots, x_n)^2 > 0$

$\iff \quad \exists a_1, \ldots, a_n \in \{-1, 1\}\ \neg\forall x_1, \ldots, x_n \in \mathbb{N}\colon Q(a_1 x_1, \ldots, a_n x_n)^2 > 0$

$\in \mathbb{Z}[x_1, \ldots, x_n]$

Sufficient Condition

all coefficients are non-negative and constant is positive (absolute positiveness)

Questions

1. how to find suitable polynomials ?
2. how to show that $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$?

Modern Approach

(a) choose abstract polynomial interpretations (linear, quadratic, . . .)

(b) transform rewrite rules into polynomial ordering constraints

(c) add monotonicity and well-definedness constraints

(d) eliminate universally quantified variables using absolute positiveness

(e) translate resulting diophantine constraints to SAT or SMT problem

Example

- rewrite system

$$\begin{aligned} 0 + y &\to y \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \end{aligned}$$

- interpretations

$$\begin{aligned} 0_{\mathcal{A}} &= 0 \\ \mathsf{s}_{\mathcal{A}}(x) &= x + 1 \\ +_{\mathcal{A}}(x, y) &= 2x + y + 1 \end{aligned}$$

- diophantine constraints $\quad \forall x, y \in \mathbb{N}$

$$e - 1 \geqslant 0 \quad da + f > 0$$
$$e - be \geqslant 0 \quad dc + f - bf - c > 0$$
$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Puzzle

$$\begin{aligned} f(0) &\to 0 & s(s(0)) &\to f(0) \\ f(s(0)) &\to s(0) & s(s(s(0))) &\to f(s(0)) \\ f(s(s(0))) &\to s(s(s(s(s(0))))) & s(s(s(s(s(s(s(0))))))) &\to f(s(s(0))) \end{aligned}$$

polynomially terminating over $\mathbb{N}$?

Outline

Definitions

- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_0 = \{x \in \mathbb{R} \mid x \geqslant 0\}$
- relation $>_{\mathbb{R},\delta}$ on $\mathbb{R}$ for every $\delta \in \mathbb{R}^+$: $x >_{\mathbb{R},\delta} y$ if $x - y \geqslant \delta$
- given $\mathcal{F}$-algebra $\mathcal{A} = (\mathbb{R}_0, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with $\delta \in \mathbb{R}^+$:

$$s >_\delta t \text{ if } [\alpha]_\mathcal{A}(s) >_{\mathbb{R},\delta} [\alpha]_\mathcal{A}(t) \text{ for all assignments } \alpha$$

Lemma

for every $\mathcal{F}$-algebra $\mathcal{A} = (\mathbb{R}_0, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ with $\delta \in \mathcal{R}^+$

- $>_\delta$ *is well-founded order*
- $>_\delta$ *is closed under substitutions*

Remark

$>_\delta$ need not be closed under contexts, even for polynomial interpretations with positive coefficients

Example

- $f_\mathcal{A}(x) = \frac{1}{2}x$ is not strictly monotone for any $\delta \in \mathbb{R}^+$:

$$f_\mathcal{A}(x+\delta) - f_\mathcal{A}(x) = \tfrac{1}{2}\delta < \delta$$

- $g_\mathcal{A}(x) = x^2$ is only strictly monotone for $\delta \geqslant 1$:

$$g_\mathcal{A}(x+\delta) - g_\mathcal{A}(x) = 2\delta x + \delta^2 \geqslant \delta \quad \Longleftrightarrow \quad \delta \geqslant 1$$

Definition

TRS $\mathcal{R}$ is polynomially terminating over $\mathbb{R}$ if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded monotone algebra $(\mathcal{A}, >)$ such that

$>_{\mathcal{A}} = >_{\delta}$

- carrier of $\mathcal{A}$ is $\mathbb{R}_0$
- $>$ is order $>_{\mathbb{R},\delta}$ restricted to $\mathbb{R}_0$ for some $\delta \in \mathbb{R}^+$
- $f_{\mathcal{A}} \in \mathbb{R}[x_1, \ldots, x_n]$ for every n-ary f

Sufficient Condition for Strict Monotonicity

1. $f_{\mathcal{A}}(x_1, \ldots, x_i + \delta, \ldots, x_n) - f_{\mathcal{A}}(x_1, \ldots, x_i, \ldots, x_n) \geqslant \delta$
2. $\dfrac{\partial f_{\mathcal{A}}(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i} \geqslant 0$

for all $x_1, \ldots, x_n \in \mathbb{R}_0$ and $1 \leqslant i \leqslant n$

Weaker Sufficient Condition for Strict Monotonicity

$$\frac{\partial f_{\mathcal{A}}(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i} \geqslant 1 \text{ for all } x_1, \ldots, x_n \in \mathbb{R}_0 \text{ and } 1 \leqslant i \leqslant n$$

Remarks

- equivalent to previous condition for linear polynomial interpretations
- independent of choice of δ
- for finite rewrite systems $>_{\mathbb{R}}$ can be used instead of $>_{\mathbb{R},\delta}$

Notation

$P_{\ell,r} \in \mathbb{R}[x_1, \ldots, x_n]$ denotes polynomial associated with rewrite rule $\ell \to r$

Lemma

$\ell >_\delta r$ *for some* $\delta \in \mathbb{R}^+ \quad \Longleftrightarrow \quad P_{\ell,r} >_{\mathbb{R}} 0$

Theorem

polynomial termination over $\mathbb{R}$ $\not\Longrightarrow$ *polynomial termination over* $\mathbb{Q}$

Proof

$\mathcal{R}$	interpretations
$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$	$\mathsf{f}_{\mathcal{A}}(x) = \sqrt{2}x + 1$
$\mathsf{g}(\mathsf{h}(x)) \to \mathsf{f}(\mathsf{h}(\mathsf{f}(x)))$	$\mathsf{g}_{\mathcal{A}}(x) = 2x$
$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) \to \mathsf{i}(x,x,x)$	$\mathsf{h}_{\mathcal{A}}(x) = x + 5$
$\mathsf{i}(\mathsf{a}_1,x,\mathsf{b}_1) \to \mathsf{i}(x,\mathsf{a}_2,\mathsf{b}_1)$	$\mathsf{i}_{\mathcal{A}}(x,y,z) = x + y + z$
$\mathsf{i}(x,\mathsf{a}_3,\mathsf{b}_1) \to \mathsf{i}(\mathsf{a}_4,x,\mathsf{b}_1)$	$\mathsf{a}_{1\mathcal{A}} = \mathsf{a}_{3\mathcal{A}} = \mathsf{b}_{1\mathcal{A}} = \mathsf{b}_{3\mathcal{A}} = 1$
$\mathsf{i}(x,x,\mathsf{b}_1) \to \mathsf{i}(\mathsf{g}(x),\mathsf{b}_2,\mathsf{b}_2)$	$\mathsf{a}_{2\mathcal{A}} = \mathsf{a}_{4\mathcal{A}} = \mathsf{b}_{2\mathcal{A}} = \mathsf{b}_{4\mathcal{A}} = 0$
$\mathsf{i}(\mathsf{g}(x),\mathsf{b}_3,\mathsf{b}_3) \to \mathsf{i}(x,x,\mathsf{b}_4)$	

$\mathcal{R}$ is not polynomially terminating over $\mathbb{Q}$

Theorem

polynomial termination over $\mathbb{Q}$ $\quad \not\Longrightarrow \quad$ *polynomial termination over* $\mathbb{N}$

Proof

$$\mathcal{R}$$

$$\begin{aligned}
\mathsf{f}(\mathsf{g}(x)) &\to \mathsf{h}(x) & \mathsf{h}(\mathsf{i}(x)) &\to \mathsf{f}(\mathsf{i}(\mathsf{g}(x))) \\
\mathsf{f}(\mathsf{f}(x)) &\to \mathsf{j}(x,x,x) & \mathsf{g}(\mathsf{g}(\mathsf{g}(x))) &\to \mathsf{j}(x,x,x) \\
\mathsf{f}(\mathsf{f}(\mathsf{f}(x))) &\to \mathsf{k}(x,x,x,x) & \mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{g}(x)))) &\to \mathsf{k}(x,x,x,x) \\
\mathsf{j}(\mathsf{a}_1,x,\mathsf{b}_1) &\to \mathsf{j}(x,\mathsf{a}_2,\mathsf{b}_1) & \mathsf{j}(x,x,\mathsf{b}_1) &\to \mathsf{j}(\mathsf{f}(x),\mathsf{b}_2,\mathsf{b}_2) \\
\mathsf{j}(x,\mathsf{a}_3,\mathsf{b}_1) &\to \mathsf{j}(\mathsf{a}_4,x,\mathsf{b}_1) & \mathsf{j}(\mathsf{f}(x),\mathsf{b}_3,\mathsf{b}_3) &\to \mathsf{j}(x,x,\mathsf{b}_4) \\
\mathsf{k}(\mathsf{a}_1,\mathsf{a}_2,x,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_1,x,\mathsf{a}_6,\mathsf{b}_1) & \mathsf{k}(x,\mathsf{a}_3,\mathsf{a}_5,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_4,x,\mathsf{a}_5,\mathsf{b}_1) \\
\mathsf{k}(\mathsf{a}_1,x,\mathsf{a}_6,\mathsf{b}_1) &\to \mathsf{k}(x,\mathsf{a}_2,\mathsf{a}_6,\mathsf{b}_1) & \mathsf{k}(\mathsf{a}_4,x,\mathsf{a}_5,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_4,\mathsf{a}_3,x,\mathsf{b}_1) \\
\mathsf{k}(x,x,x,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{h}(x),\mathsf{b}_2,\mathsf{b}_2,\mathsf{b}_2) & \mathsf{k}(\mathsf{h}(x),\mathsf{b}_3,\mathsf{b}_3,\mathsf{b}_3) &\to \mathsf{k}(x,x,x,\mathsf{b}_4)
\end{aligned}$$

$\mathcal{R}$ is polynomially terminating over $\mathbb{Q}$ but not over $\mathbb{N}$

Outline

- Well-Founded Monotone Algebras
- Polynomial Interpretations over $\mathbb{N}$
- Polynomial Interpretations over $\mathbb{R}$
- Matrix Interpretations over $\mathbb{N}$
  - String Rewriting
  - Term Rewriting
- Matrix Interpretations over $\mathbb{N} \cup \{-\infty\}$
- Further Reading

Definition

algebra $\mathcal{M}$ with proper order $>$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$
- interpretations

$$f_{\mathcal{M}} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix}$$

  with $a_{11} > 0$ and $a_{12}, \ldots, a_{dd}, b_1, \ldots, b_d \geqslant 0$

- proper order

$$\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} > \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \iff x_1 > y_1 \ \wedge \ \bigwedge_{i=2}^{d} x_i \geqslant y_i$$

Lemma

$(\mathcal{M}, >)$ *is well-founded monotone algebra*

Example

$$\begin{aligned} a(a(x)) &\to b(c(x)) \\ b(b(x)) &\to a(c(x)) \\ c(c(x)) &\to a(b(x)) \end{aligned}$$

$$a_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$b_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\forall \vec{x} \quad \begin{aligned} a_{\mathcal{M}}(a_{\mathcal{M}}(\vec{x})) &> b_{\mathcal{M}}(c_{\mathcal{M}}(\vec{x})) \\ b_{\mathcal{M}}(b_{\mathcal{M}}(\vec{x})) &> a_{\mathcal{M}}(c_{\mathcal{M}}(\vec{x})) \\ c_{\mathcal{M}}(c_{\mathcal{M}}(\vec{x})) &> a_{\mathcal{M}}(b_{\mathcal{M}}(\vec{x})) \end{aligned}$$

$$c_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Outline

Definition

algebra $\mathcal{M}$ with well-founded order $>$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$
- interpretations (for every n-ary f)

$$f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = M_1 \vec{x}_1 + \cdots + M_n \vec{x}_n + \vec{f}$$

  with

  - matrices $M_1, \ldots, M_n \in \mathbb{N}^{d \times d}$ with $(M_i)_{1,1} \geqslant 1$ for all $1 \leqslant i \leqslant n$
  - vector $\vec{f} \in \mathbb{N}^d$

- $(x_1, \ldots, x_d)^{\mathsf{T}} > (y_1, \ldots, y_d)^{\mathsf{T}} \iff x_1 > y_1 \ \wedge \ \bigwedge_{i=2}^{d} x_i \geqslant y_i$

Lemma

$(\mathcal{M}, >)$ *is well-founded monotone algebra*

Example

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{c}) \qquad \mathsf{h}(\mathsf{c}) \to \mathsf{h}(\mathsf{a})$$

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{a}_{\mathcal{M}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{b}_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathsf{h}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{c}_{\mathcal{M}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Example

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x) \qquad \mathsf{i}(\mathsf{a}_1, x, \mathsf{b}_1) \to \mathsf{i}(x, \mathsf{a}_2, \mathsf{b}_1)$$

$$\mathsf{g}(\mathsf{h}(x)) \to \mathsf{f}(\mathsf{h}(\mathsf{f}(x))) \qquad \mathsf{i}(x, \mathsf{a}_3, \mathsf{b}_1) \to \mathsf{i}(\mathsf{a}_4, x, \mathsf{b}_1)$$

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) \to \mathsf{i}(x, x, x) \qquad \mathsf{i}(x, x, \mathsf{b}_1) \to \mathsf{i}(\mathsf{g}(x), \mathsf{b}_2, \mathsf{b}_2)$$

$$\mathsf{i}(\mathsf{g}(x), \mathsf{b}_3, \mathsf{b}_3) \to \mathsf{i}(x, x, \mathsf{b}_4)$$

$$\mathsf{f}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad {\mathsf{a}_1}_{\mathcal{A}} = {\mathsf{a}_3}_{\mathcal{A}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad {\mathsf{b}_1}_{\mathcal{A}} = \begin{pmatrix} 13 \\ 0 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \vec{x} \qquad {\mathsf{a}_2}_{\mathcal{A}} = {\mathsf{a}_4}_{\mathcal{A}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad {\mathsf{b}_2}_{\mathcal{A}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\mathsf{h}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad {\mathsf{b}_3}_{\mathcal{A}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad {\mathsf{b}_4}_{\mathcal{A}} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$\mathsf{i}_{\mathcal{A}}(\vec{x}, \vec{y}, \vec{z}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Outline

- Well-Founded Monotone Algebras
- Polynomial Interpretations over $\mathbb{N}$
- Polynomial Interpretations over $\mathbb{R}$
- Matrix Interpretations over $\mathbb{N}$
  - String Rewriting
  - Term Rewriting
- Matrix Interpretations over $\mathbb{N} \cup \{-\infty\}$
  - String Rewriting
  - Term Rewriting
- Further Reading

Definitions

$\mathbb{A}_{\mathbb{N}} = \mathbb{N} \cup \{-\infty\}$ with

- well-founded order $\cdots > 1 > 0 > -\infty$
- operations
  - $x \oplus y = \max(x, y)$
  - $x \otimes y = x + y \qquad -\infty \otimes x = x \otimes -\infty = -\infty$
- $x \gg y$ if $x > y$ or $x = y = -\infty$

algebra $\mathcal{M}$ with well-founded order $\gg$

- carrier of $\mathcal{M}$ is $\mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}$ with $d > 0$
- $(x_1, \ldots, x_d)^{\mathsf{T}} \gg (y_1, \ldots, y_d)^{\mathsf{T}} \iff \bigwedge_{i=1}^{d} x_i \gg y_i$
- interpretations

$$f_{\mathcal{M}}(\vec{x}) = M \otimes \vec{x}$$

with $M \in \mathbb{A}_{\mathbb{N}}^{d \times d}$ and $M_{1,1} \geqslant 0$

Example

$$\mathsf{a}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(\mathsf{a}(x)))$$
$$\mathsf{a}(\mathsf{a}(\mathsf{a}(x))) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(x)))$$
$$\mathsf{b}(\mathsf{b}(\mathsf{b}(x))) \to \mathsf{a}(x)$$

1. $\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \vec{x} \qquad \mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 1 & -\infty \end{pmatrix} \otimes \vec{x}$
2. $\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \vec{x} \qquad \mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 0 & -\infty \\ 0 & -\infty \end{pmatrix} \otimes \vec{x}$
3. $\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \otimes \vec{x} \qquad \mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & -\infty \end{pmatrix} \otimes \vec{x}$

Outline

- Well-Founded Monotone Algebras
- Polynomial Interpretations over $\mathbb{N}$
- Polynomial Interpretations over $\mathbb{R}$
- Matrix Interpretations over $\mathbb{N}$
  - String Rewriting
  - Term Rewriting
- Matrix Interpretations over $\mathbb{N} \cup \{-\infty\}$
  - String Rewriting
  - Term Rewriting
- Further Reading

Remark

constants c (with $c_{\mathcal{M}} \in \mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}$) are allowed

Problem

no function

$$f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = M_1 \otimes \vec{x}_1 \oplus \cdots \oplus M_n \otimes \vec{x}_n$$

with $n > 1$ is strictly monotone in both arguments

Remark

every function

$$f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = M_1 \otimes \vec{x}_1 \oplus \cdots \oplus M_n \otimes \vec{x}_n$$

is weakly monotone (with respect to $>$) in all arguments

Outline

- Well-Founded Monotone Algebras
- Polynomial Interpretations over $\mathbb{N}$
- Polynomial Interpretations over $\mathbb{R}$
- Matrix Interpretations over $\mathbb{N}$
- Matrix Interpretations over $\mathbb{N} \cup \{-\infty\}$
- Further Reading

- Testing Positiveness of Polynomials
  Hoon Hong and Dalibor Jakuš
  JAR 21(1), pp. 23 – 38, 2004

- On the Relative Power of Polynomials with Real, Rational, and Integer Coefficients in Proofs of Termination of Rewriting
  Salvador Lucas
  AAECC 17(1), pp. 49 – 73, 2006

- Polynomials over the Reals in Proofs of Termination: From Theory to Practice
  Salvador Lucas
  RAIRO TIA 39, pp. 547 – 586, 2005

- Matrix Interpretations for Proving Termination of Term Rewriting
  Jörg Endrullis, Johannes Waldmann and Hans Zantema
  JAR 40(2,3), pp. 195 – 220, 2008

- Arctic Termination . . . Below Zero
  Adam Koprowski and Johannes Waldmann
  Proc. 19th RTA, LNCS 5117, pp. 202 – 216, 2008