

Advanced Topics in Termination

ISR 2009 – lecture 1

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Definition

rewrite system is **terminating** if there are no infinite rewrite sequences

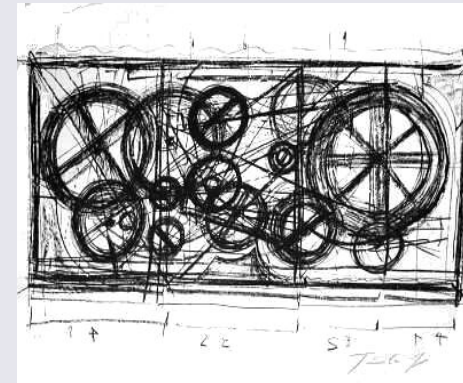
Termination Methods

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, ...

Example (Addition on Natural Numbers)

signature	0 1 ... 9 (constants)	+ :	(binary, infix)
terms	1 + 3	2 + (7 : 3)	(2 : (3 : x)) + ((1 + 7) : 2)
rewrite rules	0 + 0 → 0	1 + 0 → 1	... 9 + 0 → 9
	0 + 1 → 1	1 + 1 → 2	... 9 + 1 → 1 : 0
	0 + 2 → 2	1 + 2 → 3	... 9 + 2 → 1 : 1
	0 + 3 → 3	1 + 3 → 4	... 9 + 3 → 1 : 2
	0 + 4 → 4	1 + 4 → 5	... 9 + 4 → 1 : 3
	0 + 5 → 5	1 + 5 → 6	... 9 + 5 → 1 : 4
	0 + 6 → 6	1 + 6 → 7	... 9 + 6 → 1 : 5
	0 + 7 → 7	1 + 7 → 8	... 9 + 7 → 1 : 6
	0 + 8 → 8	1 + 8 → 9	... 9 + 8 → 1 : 7
	0 + 9 → 9	1 + 9 → 1 : 0	... 9 + 9 → 1 : 8
	x + (y : z) → y : (x + z)	0 : x → x	
	(x : y) + z → x : (y + z)	x : (y : z) → (x + y) : z	
rewriting	(2 : 3) + (7 : 7)	→ *	(1 : 0) : 0

Termination Research



Termination Tools

CiME, T_TT, AProVE, Termptation, Cariboo, Matchbox, Torpa, Teparla, Jambox, TPA, MultumNonMultum, MuTerm, T_TTbox, NTI, T_TT₂, Nonloop, TrafO, VMTL, ...

Outline

- Well-Founded Monotone Algebras
- Polynomial Interpretations over \mathbb{N}
- Polynomial Interpretations over \mathbb{R}
- Matrix Interpretations over \mathbb{N}
- Matrix Interpretations over $\mathbb{N} \cup \{-\infty\}$
- Further Reading

Topics

- polynomial interpretations
- matrix interpretations
- dependency pairs
- match-bounds
- semantic labeling
- derivational complexity
- decidable classes
- certification
- applications

Further Reading

- <http://cl-informatik.uibk.ac.at/~ami/09isr/>

Definitions

- **rewrite order** is proper order $>$ on terms which is
 - closed under contexts $s > t \implies C[s] > C[t]$
 - closed under substitutions $s > t \implies s\sigma > t\sigma$
- **reduction order** is well-founded rewrite order

Notation

$\mathcal{R} \subseteq >$ abbreviates $\ell > r$ for all rules $\ell \rightarrow r$ in \mathcal{R}

Theorem

$TRS \mathcal{R}$ is terminating $\iff \mathcal{R} \subseteq >$ for reduction order $>$

Definitions

- **well-founded monotone \mathcal{F} -algebra** $(\mathcal{A}, >)$ consists of nonempty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) > f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

for all $a_1, \dots, a_n, b \in A$ and $i \in \{1, \dots, n\}$ with $a_i > b$

- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemma

$>_{\mathcal{A}}$ is reduction order for every well-founded monotone algebra $(\mathcal{A}, >)$

Theorem

$TRS \mathcal{R}$ is terminating $\iff \mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded monotone algebra $(\mathcal{A}, >)$

Well-Founded Monotone Algebras

used in termination proofs/tools:

- polynomial interpretations over \mathbb{N}
- polynomial interpretations over \mathbb{Q} and \mathbb{R}
- matrix interpretations over \mathbb{N}
- matrix interpretations over $\mathbb{N} \cup \{-\infty\}$
- ...

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Example

- rewrite system

$$\begin{aligned} 0 + y &\rightarrow y \\ s(x) + y &\rightarrow s(x + y) \\ 0 \times y &\rightarrow 0 \\ s(x) \times y &\rightarrow y + (x \times y) \end{aligned}$$

- interpretations

$$\begin{aligned} 0_{\mathcal{A}} &= 1 \\ s_{\mathcal{A}}(x) &= x + 1 \\ +_{\mathcal{A}}(x, y) &= 2x + y \\ \times_{\mathcal{A}}(x, y) &= 2xy + x + y + 1 \end{aligned}$$

- constraints $\forall x, y \in \mathbb{N}$

$$\begin{aligned} y + 2 &> y \\ 2x + y + 2 &> 2x + y + 1 \\ 3y + 2 &> 1 \\ 2xy + x + 3y + 2 &> 2xy + x + 3y + 1 \end{aligned}$$

Definition

TRS \mathcal{R} is **polynomially terminating over \mathbb{N}** if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded **monotone** algebra $(\mathcal{A}, >)$ such that

- carrier of \mathcal{A} is \mathbb{N}
- $>$ is standard order on \mathbb{N}
- $f_{\mathcal{A}} \in \mathbb{Z}[x_1, \dots, x_n]$ for every n -ary f

polynomials with coefficients in \mathbb{Z} and indeterminates x_1, \dots, x_n

Lemma

\mathcal{R} is polynomially terminating over \mathbb{N}

\iff

\mathcal{R} is polynomially terminating over $\{n \in \mathbb{N} \mid n \geq N\}$ for some $N \geq 0$

Questions

- 1 how to find suitable polynomials ?
- 2 how to show that $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$?

Theorem

following problem is undecidable:

instance: polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$

question: $\forall x_1, \dots, x_n \in \mathbb{N}: P(x_1, \dots, x_n) > 0$?

Proof

reduction from Hilbert's 10th Problem

for arbitrary polynomial $Q \in \mathbb{Z}[x_1, \dots, x_n]$

$\exists x_1, \dots, x_n \in \mathbb{Z}: Q(x_1, \dots, x_n) = 0$

$\iff \neg \forall x_1, \dots, x_n \in \mathbb{Z}: Q(x_1, \dots, x_n) \neq 0$

$\iff \neg \forall x_1, \dots, x_n \in \mathbb{Z}: Q(x_1, \dots, x_n)^2 > 0$

$\iff \exists a_1, \dots, a_n \in \{-1, 1\} \neg \forall x_1, \dots, x_n \in \mathbb{N}: Q(a_1x_1, \dots, a_nx_n)^2 > 0$

$\in \mathbb{Z}[x_1, \dots, x_n]$

Sufficient Condition

all coefficients are non-negative and constant is positive (absolute positiveness)

Example

- rewrite system

$$\begin{aligned} 0 + y &\rightarrow y \\ s(x) + y &\rightarrow s(x + y) \end{aligned}$$

- interpretations

$$\begin{aligned} 0_{\mathcal{A}} &= 0 \\ s_{\mathcal{A}}(x) &= x + 1 \\ +_{\mathcal{A}}(x, y) &= 2x + y + 1 \end{aligned}$$

- diophantine constraints $\forall x, y \in \mathbb{N}$

$$\begin{aligned} e - 1 &\geq 0 & da + f &> 0 \\ e - be &\geq 0 & dc + f - bf - c &> 0 \\ a &\geq 0 & b &\geq 1 & c &\geq 0 & d &\geq 1 & e &\geq 1 & f &\geq 0 \end{aligned}$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Questions

- 1 how to find suitable polynomials ?
- 2 how to show that $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$?

Modern Approach

- (a) choose abstract polynomial interpretations (linear, quadratic, ...)
- (b) transform rewrite rules into polynomial ordering constraints
- (c) add monotonicity and well-definedness constraints
- (d) eliminate universally quantified variables using absolute positiveness
- (e) translate resulting diophantine constraints to SAT or SMT problem

Puzzle

$$\begin{aligned} f(0) &\rightarrow 0 & s(s(0)) &\rightarrow f(0) \\ f(s(0)) &\rightarrow s(0) & s(s(s(0))) &\rightarrow f(s(0)) \\ f(s(s(0))) &\rightarrow s(s(s(s(s(0)))))) & s(s(s(s(s(s(0)))))) &\rightarrow f(s(s(0))) \end{aligned}$$

polynomially terminating over \mathbb{N} ?

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- **Polynomial Interpretations over \mathbb{R}**
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Remark

$>_\delta$ need not be closed under contexts, even for polynomial interpretations with positive coefficients

Example

- $f_{\mathcal{A}}(x) = \frac{1}{2}x$ is not strictly monotone for any $\delta \in \mathbb{R}^+$:

$$f_{\mathcal{A}}(x + \delta) - f_{\mathcal{A}}(x) = \frac{1}{2}\delta < \delta$$

- $g_{\mathcal{A}}(x) = x^2$ is only strictly monotone for $\delta \geq 1$:

$$g_{\mathcal{A}}(x + \delta) - g_{\mathcal{A}}(x) = 2\delta x + \delta^2 \geq \delta \iff \delta \geq 1$$

Definitions

- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_0 = \{x \in \mathbb{R} \mid x \geq 0\}$
- relation $>_{\mathbb{R}, \delta}$ on \mathbb{R} for every $\delta \in \mathbb{R}^+$: $x >_{\mathbb{R}, \delta} y$ if $x - y \geq \delta$
- given \mathcal{F} -algebra $\mathcal{A} = (\mathbb{R}_0, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with $\delta \in \mathbb{R}^+$:

$s >_\delta t$ if $[\alpha]_{\mathcal{A}}(s) >_{\mathbb{R}, \delta} [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemma

for every \mathcal{F} -algebra $\mathcal{A} = (\mathbb{R}_0, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ with $\delta \in \mathbb{R}^+$

- $>_\delta$ is well-founded order
- $>_\delta$ is closed under substitutions

Definition

TRS \mathcal{R} is **polynomially terminating over \mathbb{R}** if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of \mathcal{A} is \mathbb{R}_0
- $>$ is order $>_{\mathbb{R}, \delta}$ restricted to \mathbb{R}_0 for some $\delta \in \mathbb{R}^+$
- $f_{\mathcal{A}} \in \mathbb{R}[x_1, \dots, x_n]$ for every n -ary f

$>_{\mathcal{A}} = >_\delta$

Sufficient Condition for Strict Monotonicity

1 $f_{\mathcal{A}}(x_1, \dots, x_i + \delta, \dots, x_n) - f_{\mathcal{A}}(x_1, \dots, x_i, \dots, x_n) \geq \delta$

2 $\frac{\partial f_{\mathcal{A}}(x_1, \dots, x_i, \dots, x_n)}{\partial x_i} \geq 0$

for all $x_1, \dots, x_n \in \mathbb{R}_0$ and $1 \leq i \leq n$

Weaker Sufficient Condition for Strict Monotonicity

$$\frac{\partial f_{\mathcal{A}}(x_1, \dots, x_i, \dots, x_n)}{\partial x_i} \geq 1 \text{ for all } x_1, \dots, x_n \in \mathbb{R}_0 \text{ and } 1 \leq i \leq n$$

Remarks

- equivalent to previous condition for linear polynomial interpretations
- independent of choice of δ
- for **finite** rewrite systems $>_{\mathbb{R}}$ can be used instead of $>_{\mathbb{R}, \delta}$

Notation

$P_{\ell, r} \in \mathbb{R}[x_1, \dots, x_n]$ denotes polynomial associated with rewrite rule $\ell \rightarrow r$

Lemma

$$\ell >_{\delta} r \text{ for some } \delta \in \mathbb{R}^+ \iff P_{\ell, r} >_{\mathbb{R}} 0$$

Theorem

$$\text{polynomial termination over } \mathbb{Q} \not\Rightarrow \text{polynomial termination over } \mathbb{N}$$

Proof

$\begin{aligned} f(g(x)) &\rightarrow h(x) \\ f(f(x)) &\rightarrow j(x, x, x) \\ f(f(f(x))) &\rightarrow k(x, x, x, x) \\ j(a_1, x, b_1) &\rightarrow j(x, a_2, b_1) \\ j(x, a_3, b_1) &\rightarrow j(a_4, x, b_1) \\ k(a_1, a_2, x, b_1) &\rightarrow k(a_1, x, a_6, b_1) \\ k(a_1, x, a_6, b_1) &\rightarrow k(x, a_2, a_6, b_1) \\ k(x, x, x, b_1) &\rightarrow k(h(x), b_2, b_2, b_2) \end{aligned}$	$\begin{aligned} h(i(x)) &\rightarrow f(i(g(x))) \\ g(g(g(x))) &\rightarrow j(x, x, x) \\ g(g(g(g(x)))) &\rightarrow k(x, x, x, x) \\ j(x, x, b_1) &\rightarrow j(f(x), b_2, b_2) \\ j(f(x), b_3, b_3) &\rightarrow j(x, x, b_4) \\ k(x, a_3, a_5, b_1) &\rightarrow k(a_4, x, a_5, b_1) \\ k(a_4, x, a_5, b_1) &\rightarrow k(a_4, a_3, x, b_1) \\ k(h(x), b_3, b_3, b_3) &\rightarrow k(x, x, x, b_4) \end{aligned}$
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\mathcal{R} is polynomially terminating over \mathbb{Q} but not over \mathbb{N}

Theorem

$$\text{polynomial termination over } \mathbb{R} \not\Rightarrow \text{polynomial termination over } \mathbb{Q}$$

Proof

\mathcal{R}	interpretations
$f(f(x)) \rightarrow g(x)$	$f_{\mathcal{A}}(x) = \sqrt{2}x + 1$
$g(h(x)) \rightarrow f(h(f(x)))$	$g_{\mathcal{A}}(x) = 2x$
$f(f(f(f(x)))) \rightarrow i(x, x, x)$	$h_{\mathcal{A}}(x) = x + 5$
$i(a_1, x, b_1) \rightarrow i(x, a_2, b_1)$	$i_{\mathcal{A}}(x, y, z) = x + y + z$
$i(x, a_3, b_1) \rightarrow i(a_4, x, b_1)$	$a_{1, \mathcal{A}} = a_{3, \mathcal{A}} = b_{1, \mathcal{A}} = b_{3, \mathcal{A}} = 1$
$i(x, x, b_1) \rightarrow i(g(x), b_2, b_2)$	$a_{2, \mathcal{A}} = a_{4, \mathcal{A}} = b_{2, \mathcal{A}} = b_{4, \mathcal{A}} = 0$
$i(g(x), b_3, b_3) \rightarrow i(x, x, b_4)$	

\mathcal{R} is not polynomially terminating over \mathbb{Q}

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Definition

algebra \mathcal{M} with proper order $>$

- carrier of \mathcal{M} is \mathbb{N}^d with $d > 0$
- interpretations

$$f_{\mathcal{M}} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix}$$

with $a_{11} > 0$ and $a_{12}, \dots, a_{dd}, b_1, \dots, b_d \geq 0$

- proper order

$$\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} > \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \iff x_1 > y_1 \wedge \bigwedge_{i=2}^d x_i \geq y_i$$

Lemma

$(\mathcal{M}, >)$ is well-founded monotone algebra

Example

$$\begin{aligned} a(a(x)) &\rightarrow b(c(x)) \\ b(b(x)) &\rightarrow a(c(x)) \\ c(c(x)) &\rightarrow a(b(x)) \end{aligned}$$

$$a_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$b_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \forall \vec{x} \quad & a_{\mathcal{M}}(a_{\mathcal{M}}(\vec{x})) > b_{\mathcal{M}}(c_{\mathcal{M}}(\vec{x})) \\ & b_{\mathcal{M}}(b_{\mathcal{M}}(\vec{x})) > a_{\mathcal{M}}(c_{\mathcal{M}}(\vec{x})) \\ & c_{\mathcal{M}}(c_{\mathcal{M}}(\vec{x})) > a_{\mathcal{M}}(b_{\mathcal{M}}(\vec{x})) \end{aligned}$$

$$c_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

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Definition

algebra \mathcal{M} with well-founded order $>$

- carrier of \mathcal{M} is \mathbb{N}^d with $d > 0$
- interpretations (for every n -ary f)

$$f_{\mathcal{M}}(\vec{x}_1, \dots, \vec{x}_n) = M_1 \vec{x}_1 + \dots + M_n \vec{x}_n + \vec{f}$$

with

- matrices $M_1, \dots, M_n \in \mathbb{N}^{d \times d}$ with $(M_i)_{1,1} \geq 1$ for all $1 \leq i \leq n$
- vector $\vec{f} \in \mathbb{N}^d$
- $(x_1, \dots, x_d)^T > (y_1, \dots, y_d)^T \iff x_1 > y_1 \wedge \bigwedge_{i=2}^d x_i \geq y_i$

Lemma

$(\mathcal{M}, >)$ is well-founded monotone algebra

Example

$$f(a) \rightarrow f(b) \quad g(b) \rightarrow g(c) \quad h(c) \rightarrow h(a)$$

$$f_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x}$$

$$a_{\mathcal{M}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$g_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x}$$

$$b_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$h_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x}$$

$$c_{\mathcal{M}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

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Example

$$\begin{array}{ll} f(f(x)) \rightarrow g(x) & i(a_1, x, b_1) \rightarrow i(x, a_2, b_1) \\ g(h(x)) \rightarrow f(h(f(x))) & i(x, a_3, b_1) \rightarrow i(a_4, x, b_1) \\ f(f(f(f(x)))) \rightarrow i(x, x, x) & i(x, x, b_1) \rightarrow i(g(x), b_2, b_2) \\ & i(g(x), b_3, b_3) \rightarrow i(x, x, b_4) \end{array}$$

$$f_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad a_{1,\mathcal{A}} = a_{3,\mathcal{A}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad b_{1,\mathcal{A}} = \begin{pmatrix} 13 \\ 0 \end{pmatrix}$$

$$g_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \vec{x} \quad a_{2,\mathcal{A}} = a_{4,\mathcal{A}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b_{2,\mathcal{A}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$h_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad b_{3,\mathcal{A}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad b_{4,\mathcal{A}} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$i_{\mathcal{A}}(\vec{x}, \vec{y}, \vec{z}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Definitions

$\mathbb{A}_{\mathbb{N}} = \mathbb{N} \cup \{-\infty\}$ with

- well-founded order $\dots > 1 > 0 > -\infty$
- operations
 - $x \oplus y = \max(x, y)$
 - $x \otimes y = x + y \quad -\infty \otimes x = x \otimes -\infty = -\infty$
 - $x \gg y$ if $x > y$ or $x = y = -\infty$

algebra \mathcal{M} with **well-founded order** \gg

- carrier of \mathcal{M} is $\mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}$ with $d > 0$
- $(x_1, \dots, x_d)^T \gg (y_1, \dots, y_d)^T \iff \bigwedge_{i=1}^d x_i \gg y_i$
- interpretations

$$f_{\mathcal{M}}(\vec{x}) = M \otimes \vec{x}$$

with $M \in \mathbb{A}_{\mathbb{N}}^{d \times d}$ and $M_{1,1} \geq 0$

Example

$$a(a(x)) \rightarrow a(b(a(x)))$$

$$a(a(a(x))) \rightarrow b(a(a(x)))$$

$$b(b(b(x))) \rightarrow a(x)$$

$$1 \quad a_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \vec{x} \quad b_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 1 & -\infty \end{pmatrix} \otimes \vec{x}$$

$$2 \quad a_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \vec{x} \quad b_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 0 & -\infty \\ 0 & -\infty \end{pmatrix} \otimes \vec{x}$$

$$3 \quad a_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \otimes \vec{x} \quad b_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & -\infty \end{pmatrix} \otimes \vec{x}$$

Remark

constants c (with $c_{\mathcal{M}} \in \mathbb{N} \times \mathbb{A}_{\mathbb{N}}^{d-1}$) are allowed

Problem

no function

$$f_{\mathcal{M}}(\vec{x}_1, \dots, \vec{x}_n) = M_1 \otimes \vec{x}_1 \oplus \dots \oplus M_n \otimes \vec{x}_n$$

with $n > 1$ is strictly monotone in both arguments

Remark

every function

$$f_{\mathcal{M}}(\vec{x}_1, \dots, \vec{x}_n) = M_1 \otimes \vec{x}_1 \oplus \dots \oplus M_n \otimes \vec{x}_n$$






is **weakly monotone** (with respect to $>$) in all arguments

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- Further Reading

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