Advanced Topics in Termination

ISR 2009 – lecture 3

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Topics

- polynomial interpretations
- matrix interpretations
- dependency pairs
- match-bounds
- semantic labeling
- derivational complexity
- decidable classes
- certification
- applications

Further Reading

- `http://cl-informatik.uibk.ac.at/~ami/09isr/`

Puzzle

$\mathcal{C}_{\mathcal{E}}$-compatible reduction pair $(>, \gtrsim)$

$$(\mathcal{P}, \mathcal{R}) \mapsto \begin{cases} \{(\mathcal{P} \setminus >, \mathcal{U}(\mathcal{P}, \mathcal{R}))\} & \text{if } \mathcal{P} \subseteq > \cup \gtrsim \text{ and } \mathcal{U}(\mathcal{P}, \mathcal{R}) \subseteq \gtrsim \\ \{(\mathcal{P}, \mathcal{R})\} & \text{otherwise} \end{cases}$$

is this DP processor sound ?

Solution

no . . .

- $\mathcal{R}$ $\quad \mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x) \qquad \mathsf{g}(x, y) \to x \qquad \mathsf{g}(x, y) \to y$
- $\mathcal{P}$ $\quad \mathsf{F}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{F}(x, x, x)$

Outline

- Match-Bounds
  - String Rewriting
  - Term Rewriting
  - Right-Hand Sides of Forward Closures
- Semantic Labeling
- Further Reading

Definition

TRS $\mathcal{S}$ over signature $\mathcal{G}$ is enrichment of TRS $\mathcal{R}$ over signature $\mathcal{F}$ if $\exists$ mappings

- lift : $\mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathcal{T}(\mathcal{G},\mathcal{V})$
- base : $\mathcal{T}(\mathcal{G},\mathcal{V}) \to \mathcal{T}(\mathcal{F},\mathcal{V})$

such that

1. $\text{base}(\text{lift}(t)) = t$ for all terms $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$
2. if $t_1 \xrightarrow[\mathcal{R}]{} t_2$ and $\text{base}(u_1) = t_1$ then $u_1 \xrightarrow[\mathcal{S}]{} u_2$ with $\text{base}(u_2) = t_2$

Lemma

enrichment $\mathcal{S}$ of $\mathcal{R}$ is terminating $\implies$ *$\mathcal{R}$ is terminating*

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is locally terminating if every restriction of $\mathcal{R}$ to finite signature $\mathcal{G} \subseteq \mathcal{F}$ is terminating

Definitions

TRS $\mathcal{R}$ over signature $\mathcal{F}$ $\qquad L \subseteq \mathcal{T}(\mathcal{F})$

- $(\xrightarrow[\mathcal{R}]{*})(L) = \{u \in \mathcal{T}(\mathcal{F}) \mid t \xrightarrow[\mathcal{R}]{*} u \text{ for some } t \in L\}$
- $\mathcal{R}$ is compact for L if $(\xrightarrow[\mathcal{R}]{*})(L) \subseteq \mathcal{T}(\mathcal{G})$ for finite signature $\mathcal{G} \subseteq \mathcal{F}$

Lemma

$\mathcal{R}$ is locally terminating and compact for $\mathcal{T}(\mathcal{F})$ $\implies$ $\mathcal{R}$ is terminating

Idea

prove termination of TRS $\mathcal{R}$ by constructing enrichment $\mathcal{S}$ such that

1. $\mathcal{S}$ is locally terminating
2. $\mathcal{S}$ is compact for $\mathcal{T}(\mathcal{F})$

Definition (TRS $\mathcal{R}$ over signature $\mathcal{F}$)

- signature $\mathcal{F}_{\mathbb{N}} = \{f_n \mid f \in \mathcal{F} \text{ and } n \in \mathbb{N}\}$
- mappings base: $\mathcal{F}_{\mathbb{N}} \to \mathcal{F}$ height: $\mathcal{F}_{\mathbb{N}} \to \mathcal{F}$ $\text{lift}_c : \mathcal{F} \to \mathcal{F}_{\mathbb{N}}$ for $c \in \mathbb{N}$

$$\text{base}(f_n) = f \qquad \text{height}(f_n) = n \qquad \text{lift}_c(f) = f_c$$

- TRS match($\mathcal{R}$) consists of all rules $l \to \text{lift}_c(r)$ such that
  1. $\text{base}(l) \to r \in \mathcal{R}$
  2. $c = 1 + \min\{\, \text{height}(l(p)) \mid p \in \mathcal{FP}\text{os}(l)\,\}$

Example

$\mathcal{R}$ $\quad \mathsf{f}(\mathsf{g}(\mathsf{a}, x), y) \to \mathsf{h}(\mathsf{f}(x, y))$

$$\begin{array}{rcl} \mathsf{f}_0(\mathsf{g}_0(\mathsf{a}_0, x), y) & \to & \mathsf{h}_1(\mathsf{f}_1(x, y)) \\ \mathsf{f}_0(\mathsf{g}_0(\mathsf{a}_1, x), y) & \to & \mathsf{h}_1(\mathsf{f}_1(x, y)) \\ \mathsf{f}_2(\mathsf{g}_1(\mathsf{a}_1, x), y) & \to & \mathsf{h}_2(\mathsf{f}_2(x, y)) \\ \mathsf{f}_7(\mathsf{g}_4(\mathsf{a}_3, x), y) & \to & \mathsf{h}_4(\mathsf{f}_4(x, y)) \\ & \cdots & \end{array} \quad \mathsf{match}(\mathcal{R})$$

Lemma

match $(\mathcal{R})$ *is enrichment of* $\mathcal{R}$ *for every left-linear TRS* $\mathcal{R}$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is match-bounded if

$$(\xrightarrow[\mathsf{match}(\mathcal{R})]{*})(\mathsf{lift}_0(\mathcal{T}(\mathcal{F}))) \subseteq \mathcal{T}(\mathcal{F}_{\{0,\ldots,c\}})$$

for some $c \in \mathbb{N}$

Outline

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- Semantic Labeling
  - Model Version
  - Quasi-Model Version
  - Root-Labeling
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Theorem

match $(\mathcal{R})$ *is locally terminating for every SRS* $\mathcal{R}$

Corollary

match-bounded SRSs are terminating

Remark

match $(\mathcal{R})$ *is not locally terminating for arbitrary TRSs* $\mathcal{R}$

Example

$\mathcal{R}$ $\quad \mathsf{f}(\mathsf{a}, x) \to \mathsf{f}(x, x)$ $\qquad \mathsf{f}_1(\mathsf{a}_0, x) \to \mathsf{f}_1(x, x)$ $\quad \cdots \quad$ match $(\mathcal{R})$

Question

how to prove match-boundedness ?

Answer

use automata techniques

Idea

SRS $\mathcal{R}$ over signature $\mathcal{F}$

1. construct finite automaton $\mathcal{A}$ that accepts $\text{lift}_0(\mathcal{T}(\mathcal{F}))$
2. close $\mathcal{A}$ under rewriting with respect to $\text{match}(\mathcal{R})$

Definition

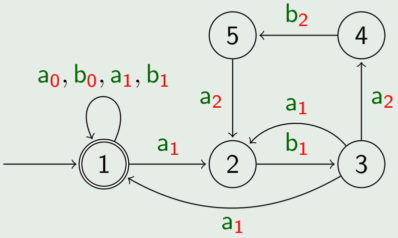
finite automaton $\mathcal{A} = (Q, \Sigma, \delta, s, F)$ is compatible with SRS $\mathcal{R}$ over signature $\mathcal{F}$ if

1. $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{L}(\mathcal{A})$
2. $\forall\, l \to r \in \mathcal{R} \quad \forall\, p, q \in Q \quad \text{if } p \xrightarrow[\mathcal{A}]{l} q \text{ then } p \xrightarrow[\mathcal{A}]{r} q$

Example

rewrite rule $\mathsf{a}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(\mathsf{a}(x)))$ match-bounded by 2

finite automaton



compatibility violations

$1 \xrightarrow{a_0} 1 \xrightarrow{a_0} 1$ $\qquad 1 \xrightarrow{a_1} 2 \xrightarrow{b_1} 3 \xrightarrow{a_1} 1$

$1 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2$ $\qquad 1 \xrightarrow{a_1} 2 \xrightarrow{b_1} 3 \xrightarrow{a_1} 2$

$3 \xrightarrow{a_1} 1 \xrightarrow{a_0} 1$ $\qquad 3 \xrightarrow{a_1} 1 \xrightarrow{b_1} 1 \xrightarrow{a_1} 1$

$3 \xrightarrow{a_1} 1 \xrightarrow{a_1} 2$ $\qquad 3 \xrightarrow{a_2} 4 \xrightarrow{b_2} 5 \xrightarrow{a_2} 2$

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Lemma

match $(\mathcal{R})$ *is locally terminating for right-linear TRSs*

Corollary

linear match-bounded TRSs are terminating

Definitions

- TRS top$(\mathcal{R})$ consists of all rules $l \to \mathrm{lift}_c(r)$ such that
  1. $\mathrm{base}(l) \to r \in \mathcal{R}$
  2. $c = 1 + \mathrm{height}(l(\epsilon))$
- TRS roof $(\mathcal{R})$ consists of all rules $l \to \mathrm{lift}_c(r)$ such that
  1. $\mathrm{base}(l) \to r \in \mathcal{R}$
  2. $c = 1 + \min\{\, \mathrm{height}(l(p)) \mid p \in \mathcal{RP}\mathrm{os}_{\mathcal{V}\mathrm{ar}(r)}(l) \,\}$
- $\mathcal{RP}\mathrm{os}_V(t) = \{\, p \in \mathcal{FP}\mathrm{os}(t) \mid V \subseteq \mathcal{V}\mathrm{ar}(t|_p) \,\}$

Example

$\mathcal{R}$ $\quad \mathsf{f}(\mathsf{g}(x,y),\mathsf{a}) \to \mathsf{h}(\mathsf{f}(x,y))$

$$\begin{aligned}
\mathsf{f}_0(\mathsf{g}_0(x,y),\mathsf{a}_0) &\to \mathsf{h}_1(\mathsf{f}_1(x,y))\\
\mathsf{f}_0(\mathsf{g}_0(x,y),\mathsf{a}_1) &\to \mathsf{h}_1(\mathsf{f}_1(x,y))\\
\mathsf{f}_2(\mathsf{g}_1(x,y),\mathsf{a}_1) &\to \mathsf{h}_3(\mathsf{f}_3(x,y))\\
\mathsf{f}_7(\mathsf{g}_4(x,y),\mathsf{a}_3) &\to \mathsf{h}_5(\mathsf{f}_5(x,y))\\
&\cdots
\end{aligned}$$

roof $(\mathcal{R})$

Lemma

- *top* $(\mathcal{R})$ *and roof* $(\mathcal{R})$ *are locally terminating*
- *top* $(\mathcal{R})$ *and roof* $(\mathcal{R})$ *are enrichments of left-linear* $\mathcal{R}$

Definition

- TRS $\mathcal{R}$ over signature $\mathcal{F}$ is top-bounded if

$$(\xrightarrow[\text{top}\,(\mathcal{R})]{*})(\text{lift}_0(\mathcal{T}(\mathcal{F}))) \subseteq \mathcal{T}(\mathcal{F}_{\{0,\ldots,c\}})$$

  for some $c \in \mathbb{N}$

- TRS $\mathcal{R}$ over signature $\mathcal{F}$ is roof-bounded if

$$(\xrightarrow[\text{roof}\,(\mathcal{R})]{*})(\text{lift}_0(\mathcal{T}(\mathcal{F}))) \subseteq \mathcal{T}(\mathcal{F}_{\{0,\ldots,c\}})$$

  for some $c \in \mathbb{N}$

Corollary

left-linear top-bounded and roof-bounded TRSs are terminating

Lemma

top-boundedness $\Longrightarrow$ *roof-boundedness* $\Longrightarrow$ *match-boundedness*

Example

rewrite rule $\mathsf{f}(\mathsf{f}(x,\mathsf{a}),\mathsf{a}) \to \mathsf{f}(x,\mathsf{f}(x,\mathsf{a}))$ roof-bounded by 1

not top-bounded

- $t_n = \begin{cases} \mathsf{a} & \text{if } n = 0 \\ \mathsf{f}(t_{n-1},\mathsf{a}) & \text{if } n > 0 \end{cases}$
- $\forall\, n \geqslant 0 \quad t_{n+2} = \mathsf{f}(\mathsf{f}(t_n,\mathsf{a}),\mathsf{a}) \ \to^{n+1} \mathsf{f}(t_n,\mathsf{f}(t_{n-1},\dots,\mathsf{f}(\mathsf{a},\mathsf{a})\dots))$

tree automaton

$$\begin{aligned} \mathsf{a}_0 &\to 0 \\ \mathsf{f}_0(0,0) &\to 0 \\ \mathsf{a}_1 &\to 1 \\ \mathsf{f}_1(0,1) &\to 2 \\ \mathsf{f}_1(0,2) &\to 0 \\ \mathsf{f}_1(0,2) &\to 2 \end{aligned}$$

compatibility violations

$$\mathsf{f}_0(\mathsf{f}_0(0,\mathsf{a}_0),\mathsf{a}_0) \to^* 0 \qquad \mathsf{f}_1(0,\mathsf{f}_1(0,\mathsf{a}_1)) \to^* 0$$

$$\mathsf{f}_1(\mathsf{f}_0(0,\mathsf{a}_0),\mathsf{a}_1) \to^* 2 \qquad \mathsf{f}_1(0,\mathsf{f}_1(0,\mathsf{a}_1)) \to^* 2$$

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Definition

RFC $(\mathcal{R})$ is least extension of $\mathrm{rhs}(\mathcal{R})$ closed under narrowing:

$t[r]_p\sigma \in \mathrm{RFC}(\mathcal{R})$ whenever $t \in \mathrm{RFC}(\mathcal{R})$ and $\exists\ p \in \mathcal{FP}\mathrm{os}(t)\ \exists\ l \to r \in \mathcal{R}$ such that σ is mgu of $t|_p$ and l

Theorem

right-linear TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R}$ is terminating on RFC$(\mathcal{R})$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is RFC-match-bounded if

$$(\xrightarrow[\mathsf{match}(\mathcal{R})]{*})(\mathrm{lift}_0(\mathrm{RFC}(\mathcal{R})\sigma_\#)) \subseteq \mathcal{T}(\mathcal{F}_{\{0,\ldots,c\}})$$

for some $c \in \mathbb{N}$

Notation

$\sigma_{\#}$ maps all variables to fresh constant $\#$

Definition

$\mathcal{R}_{\#}$ is least extension of $\mathcal{R}$ such that $l[\#]_p \to r\sigma \in \mathcal{R}_{\#}$ whenever $\exists\ l \to r \in \mathcal{R}_{\#}$ $\exists\ p \in \mathcal{FP}\mathrm{os}(l) \setminus \{\epsilon\}$ with

$$\sigma(x) = \begin{cases} \# & \text{if } x \in \mathcal{V}\mathrm{ar}(l|_p) \\ x & \text{otherwise} \end{cases}$$

Theorem

$RFC(\mathcal{R})\sigma_{\#} = (\xrightarrow[\mathcal{R}_{\#}]{*})(rhs(\mathcal{R})\sigma_{\#})$ for linear TRSs $\mathcal{R}$

Example

$\mathcal{R}$ $\quad \mathsf{f}(\mathsf{f}(x,\mathsf{a}),\mathsf{a}) \to \mathsf{f}(x,\mathsf{f}(x,\mathsf{a}))$

$$
\begin{aligned}
\mathsf{f}(\mathsf{f}(x,\mathsf{a}),\mathsf{a}) &\to \mathsf{f}(x,\mathsf{f}(x,\mathsf{a})) \\
\mathsf{f}(\mathsf{f}(x,\#),\mathsf{a}) &\to \mathsf{f}(x,\mathsf{f}(x,\mathsf{a})) \\
\mathsf{f}(\mathsf{f}(x,\mathsf{a}),\#) &\to \mathsf{f}(x,\mathsf{f}(x,\mathsf{a})) \\
\mathsf{f}(\mathsf{f}(x,\#),\#) &\to \mathsf{f}(x,\mathsf{f}(x,\mathsf{a})) \\
\mathsf{f}(\#,\mathsf{a}) &\to \mathsf{f}(\#,\mathsf{f}(\#,\mathsf{a})) \\
\mathsf{f}(\#,\#) &\to \mathsf{f}(\#,\mathsf{f}(\#,\mathsf{a}))
\end{aligned}
\qquad \mathcal{R}_\#
$$

Extensions

- efficient exact construction of $(\xrightarrow[\text{match}(\mathcal{R})]{*})(\text{lift}_0(\mathcal{T}(\mathcal{F})))$ for SRSs
- extension to non-left-linear TRSs using quasi-deterministic automata
- DP processors based on bounds

Outline

- Match-Bounds
- Semantic Labeling
  - Model Version
  - Quasi-Model Version
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Definitions (TRS $\mathcal{R}$ over signature $\mathcal{F}$)

- semantics
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$
- labeling
  - sets of labels $L_f \subseteq A$ for every $f \in \mathcal{F}$
  - labeling functions $\ell_f \colon A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \varnothing$
- labeling of terms (for every assignment $\alpha \colon \mathcal{V} \to A$)

$$\mathsf{lab}_\alpha(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f = \varnothing \\ f_a(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

with $a = \ell_f([\alpha]_\mathcal{A}(t_1), \dots, [\alpha]_\mathcal{A}(t_n))$

- labeled TRS

$$\mathcal{R}_{\mathsf{lab}} = \{\, \mathsf{lab}_\alpha(l) \to \mathsf{lab}_\alpha(r) \mid l \to r \in \mathcal{R} \text{ and } \alpha \colon \mathcal{V} \to A \,\}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$$

- algebra $\mathcal{A}$ is model of $\mathcal{R}$

$$A = \{0, 1\} \qquad \mathsf{a}_{\mathcal{A}} = 0 \qquad \mathsf{b}_{\mathcal{A}} = 1 \qquad \mathsf{f}_{\mathcal{A}}(x, y, z) = 0$$

- labeling ℓ

$$L_{\mathsf{a}} = L_{\mathsf{b}} = \varnothing \qquad L_{\mathsf{f}} = A \qquad \ell_{\mathsf{f}}(x, y, z) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

- TRS $\mathcal{R}_{\mathsf{lab}}$

$$\mathsf{f}_1(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}_0(x, x, x)$$

is terminating because it lacks dependency pairs

Theorem

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is terminating if

$\exists$ *algebra* $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$

$\exists$ *labeling* ℓ

such that

1. $\mathcal{R} \subseteq =_\mathcal{A}$ "$\mathcal{A}$ *is model of* $\mathcal{R}$"
2. $\mathcal{R}_{lab}$ *is terminating*

Theorem (rephrased)

TRS $\mathcal{R}$, model $\mathcal{A}$, labeling ℓ (semantic labeling is sound and complete)

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{lab} \text{ is terminating}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

- weakly monotone algebra $\mathcal{A}$

$$A = \{0,1,2\} \quad 2 > 0 \quad 1 > 0 \quad \mathsf{a}_{\mathcal{A}} = 1 \quad \mathsf{b}_{\mathcal{A}} = 2 \quad \mathsf{c}_{\mathcal{A}} = \mathsf{f}_{\mathcal{A}}(x,y,z) = 0$$

- weakly monotone labeling ℓ

$$L_{\mathsf{a}} = L_{\mathsf{b}} = L_{\mathsf{c}} = \varnothing \qquad L_{\mathsf{f}} = \{0,1\} \qquad \ell_{\mathsf{f}}(x,y,z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{if otherwise} \end{cases}$$

- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$

$$\begin{array}{llll} \mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x) & \mathsf{a} \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{f}_0(x,y,z) \to \mathsf{c} \\ \mathsf{f}_1(x,y,z) \to \mathsf{f}_0(x,y,z) & & & \mathsf{f}_1(x,y,z) \to \mathsf{c} \end{array}$$

is terminating because dependency graph lacks SCCs

Definitions

- weakly monotone $\mathcal{F}$-algebra $(\mathcal{A}, >, \geqslant)$ with $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ and $\geqslant \, = \, >^{=}$
- weakly monotone labeling functions $\ell_f \colon A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \varnothing$
- $\mathcal{D}\mathsf{ec} = \{\, f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid a, b \in L_f$ with $a > b \,\}$

Theorem

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is terminating if

$\exists$ *weakly monotone algebra $(\mathcal{A}, >, \geqslant)$*

$\exists$ *weakly monotone labeling ℓ*

such that

1. $\mathcal{R} \subseteq \, \geqslant_{\mathcal{A}}$ *"$\mathcal{A}$ is quasi-model of $\mathcal{R}$"*
2. $\mathcal{R}_{lab} \cup \mathcal{D}ec$ *is terminating*

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

- weakly monotone algebra $\mathcal{A}$

$$A = \{0,1,2\} \quad 2 > 0 \quad 1 > 0 \quad \mathsf{a}_{\mathcal{A}} = 1 \quad \mathsf{b}_{\mathcal{A}} = 2 \quad \mathsf{c}_{\mathcal{A}} = \mathsf{f}_{\mathcal{A}}(x,y,z) = 0$$

- weakly monotone labeling ℓ

$$L_{\mathsf{a}} = L_{\mathsf{b}} = L_{\mathsf{c}} = \varnothing \qquad L_{\mathsf{f}} = \{0,1\} \qquad \ell_{\mathsf{f}}(x,y,z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{if otherwise} \end{cases}$$

- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$

$$\begin{array}{llll} \mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x) & \mathsf{a} \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{f}_0(x,y,z) \to \mathsf{c} \\ \mathsf{f}_1(x,y,z) \to \mathsf{f}_0(x,y,z) & & & \mathsf{f}_1(x,y,z) \to \mathsf{c} \end{array}$$

is terminating because dependency graph lacks SCCs

Theorem (rephrased)

TRS $\mathcal{R}$, quasi-model $\mathcal{A}$, weakly monotone labeling ℓ

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{lab} \text{ is terminating}$$

Example

- TRS $\mathcal{R}$

$$\begin{aligned}
\lambda(x) \circ y &\to \lambda(x \circ (1 \star (y \circ \uparrow))) & \mathsf{id} \circ x &\to x & 1 \circ (x \star y) &\to x \\
(x \star y) \circ z &\to (x \circ z) \star (y \circ z) & 1 \circ \mathsf{id} &\to 1 & \uparrow \circ (x \star y) &\to y \\
(x \circ y) \circ z &\to x \circ (y \circ z) & \uparrow \circ \mathsf{id} &\to \uparrow
\end{aligned}$$

- weakly monotone algebra $\mathcal{A}$
  - carrier $A = \mathbb{N}$ with standard order $>$
  - interpretations

$$\begin{aligned}
\lambda_{\mathcal{A}}(x) &= x + 1 & \circ_{\mathcal{A}}(x, y) &= x + y \\
\star_{\mathcal{A}}(x, y) &= \max(x, y) & 1_{\mathcal{A}} = \uparrow_{\mathcal{A}} &= \mathsf{id}_{\mathcal{A}} = 0
\end{aligned}$$

Example (cont'd)

- weakly monotone labeling ℓ

$$L_\lambda = L_\star = L_1 = L_\uparrow = \varnothing \qquad L_\circ = \mathbb{N} \qquad \ell_\circ(x,y) = x + y$$

- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec} \quad \forall\, i,j,k \geqslant 0 \quad \forall\, l > m \geqslant 0$

$$\begin{aligned}
\lambda(x) \circ_{i+j+1} y &\to \lambda(x \circ_{i+j} (1 \star (y \circ_j \uparrow))) \\
(x \star y) \circ_{\max(i,j)+k} z &\to (x \circ_{i+k} z) \star (y \circ_{j+k} z) \\
(x \circ_{i+j} y) \circ_{i+j+k} z &\to x \circ_{i+j+k} (y \circ_{j+k} z)
\end{aligned}$$

$$\begin{aligned}
\mathsf{id} \circ_i x &\to x & 1 \circ_{\max(i,j)} (x \star y) &\to x \\
1 \circ_0 \mathsf{id} &\to 1 & \uparrow \circ_{\max(i,j)} (x \star y) &\to y \\
\uparrow \circ_0 \mathsf{id} &\to \uparrow & x \circ_l y &\to x \circ_m y
\end{aligned}$$

- LPO with precedence

$$\cdots > \circ_2 > \circ_1 > \circ_0 > \star, \lambda, 1, \uparrow$$

Extensions

- quasi-model condition required for usable rules only (predictive labeling)
- incorporation of argument filters
- DP processors based on semantic labeling

Questions

- how to choose (quasi-)model ?
- how to choose labeling ?

Answers

- ...
- root-labeling

Definition

- algebra $\mathcal{A}_\mathcal{F}$
  - $A_\mathcal{F} = \mathcal{F}$
  - $f_{\mathcal{A}_\mathcal{F}}(x_1, \ldots, x_n) = f \quad \forall\, n\text{-ary } f \in \mathcal{F}$
- labeling
  - $L_f = \mathcal{F}^n$
  - $\ell_f(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$

Example

$\mathcal{R}$ $\quad \mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$

$$\begin{aligned} \mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{a}(x)) &\to \mathsf{c}_\mathsf{a}(x) \\ \mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{b}(x)) &\to \mathsf{c}_\mathsf{b}(x) \\ \mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{c}(x)) &\to \mathsf{c}_\mathsf{c}(x) \end{aligned} \qquad \mathcal{R}_{\mathsf{rlab}}$$

Problem

no model: $\quad \forall\, \alpha \quad [\alpha](\mathsf{a}(\mathsf{b}(x))) = \mathsf{a} \neq \mathsf{c} = [\alpha](\mathsf{c}(x))$

Definitions (TRS $\mathcal{R}$ over signature $\mathcal{F}$)

- root-preserving rules

  $\mathcal{R}_\mathsf{p} = \{\, l \to r \in \mathcal{R} \mid \mathrm{root}(l) = \mathrm{root}(r) \,\}$

- root-altering rules

  $\mathcal{R}_\mathsf{a} = \{\, l \to r \in \mathcal{R} \mid \mathrm{root}(l) \neq \mathrm{root}(r) \,\}$

- flat contexts

  $\mathcal{FC} = \{\, g(x_1, \dots, x_{i-1}, \Box, x_i, \dots, x_n) \mid g \in \mathcal{F}$ has arity n and $1 \leqslant i \leqslant n \,\}$

- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}_\mathsf{p} \cup \{\, C[l] \to C[r] \mid l \to r \in \mathcal{R}_\mathsf{a}$ and $C \in \mathcal{FC} \,\}$

Lemma

- *$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})$ is terminating*
- *$\mathcal{A}_\mathcal{F}$ is model of $\mathcal{FC}(\mathcal{R})$*

Example

$\mathcal{R}$

$$a(b(x)) \to c(x)$$

$\mathcal{FC}(\mathcal{R})$

$$a(a(b(x))) \to a(c(x)) \qquad a(\Box)$$
$$b(a(b(x))) \to b(c(x)) \qquad b(\Box)$$
$$c(a(b(x))) \to c(c(x)) \qquad c(\Box)$$

$\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$

$$a_a(a_b(b_a(x))) \to a_c(c_a(x))$$
$$a_a(a_b(b_b(x))) \to a_c(c_b(x))$$
$$a_a(a_b(b_c(x))) \to a_c(c_c(x))$$
$$b_a(a_b(b_a(x))) \to b_c(c_a(x))$$
$$b_a(a_b(b_b(x))) \to b_c(c_b(x))$$
$$b_a(a_b(b_c(x))) \to b_c(c_c(x))$$
$$c_a(a_b(b_a(x))) \to c_c(c_a(x))$$
$$c_a(a_b(b_b(x))) \to c_c(c_b(x))$$
$$c_a(a_b(b_c(x))) \to c_c(c_c(x))$$

Corollary

$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})_{rlab}$ is terminating

Example (1)

- $\mathcal{R}$ $\quad \mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$
$$\begin{aligned} \mathsf{f}_{(\mathsf{a},\mathsf{b},\mathsf{a})}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}_{(\mathsf{a},\mathsf{a},\mathsf{a})}(x, x, x) \\ \mathsf{f}_{(\mathsf{a},\mathsf{b},\mathsf{b})}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}_{(\mathsf{b},\mathsf{b},\mathsf{b})}(x, x, x) \\ \mathsf{f}_{(\mathsf{a},\mathsf{b},\mathsf{f})}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}_{(\mathsf{f},\mathsf{f},\mathsf{f})}(x, x, x) \end{aligned}$$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ is terminating because it lacks dependency pairs

Example (2)

- $\mathcal{R}$ $\qquad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ $\qquad \mathsf{f}_{\mathsf{f}}(\mathsf{f}_{\mathsf{f}}(x)) \to \mathsf{f}_{\mathsf{g}}(\mathsf{g}_{\mathsf{f}}(\mathsf{f}_{\mathsf{f}}(x)))$
  $\mathsf{f}_{\mathsf{f}}(\mathsf{f}_{\mathsf{g}}(x)) \to \mathsf{f}_{\mathsf{g}}(\mathsf{g}_{\mathsf{f}}(\mathsf{f}_{\mathsf{g}}(x)))$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ is polynomially terminating over $\mathbb{N}$

Outline

- Match-Bounds
- Semantic Labeling
- Further Reading

On Tree Automata that Certify Termination of Left-Linear Term Rewriting Systems
Alfons Geser, Dieter Hofbauer, Johannes Waldmann and Hans Zantema
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Match-Bounds Revisited
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Termination of Term Rewriting by Semantic Labelling
Hans Zantema
FI 24(1,2), pp. 89 – 105, 1995

Innermost Termination of Rewrite Systems by Labeling
René Thiemann and Aart Middeldorp
Proc. 7th WRS, ENTCS 204, pp. 3 – 19, 2008

Root-Labeling
Christian Sternagel and Aart Middeldorp
Proc. 19th RTA, LNCS 5117, pp. 336 – 350, 2008