

Advanced Topics in Termination

ISR 2009 – lecture 3

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Puzzle

Puzzle

\mathcal{C}_g -compatible reduction pair $(>, \gtrsim)$

$$(\mathcal{P}, \mathcal{R}) \mapsto \begin{cases} \{(\mathcal{P} \setminus >, \mathcal{U}(\mathcal{P}, \mathcal{R}))\} & \text{if } \mathcal{P} \subseteq > \cup \gtrsim \text{ and } \mathcal{U}(\mathcal{P}, \mathcal{R}) \subseteq \gtrsim \\ \{(\mathcal{P}, \mathcal{R})\} & \text{otherwise} \end{cases}$$

is this DP processor sound ?

Solution

no ...

- \mathcal{R} $f(a, b, x) \rightarrow f(x, x, x)$ $g(x, y) \rightarrow x$ $g(x, y) \rightarrow y$
- \mathcal{P} $F(a, b, x) \rightarrow F(x, x, x)$

Topics

- polynomial interpretations
- matrix interpretations
- dependency pairs
- **match-bounds**
- **semantic labeling**
- derivational complexity
- decidable classes
- certification
- applications

Further Reading

- <http://cl-informatik.uibk.ac.at/~ami/09isr/>

Outline

- **Match-Bounds**
 - String Rewriting
 - Term Rewriting
 - Right-Hand Sides of Forward Closures
- Semantic Labeling
- Further Reading

Definition

TRS \mathcal{S} over signature \mathcal{G} is **enrichment** of TRS \mathcal{R} over signature \mathcal{F} if \exists mappings

- $\text{lift} : \mathcal{T}(\mathcal{F}, \mathcal{V}) \rightarrow \mathcal{T}(\mathcal{G}, \mathcal{V})$
- $\text{base} : \mathcal{T}(\mathcal{G}, \mathcal{V}) \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$

such that

- 1 $\text{base}(\text{lift}(t)) = t$ for all terms $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$
- 2 if $t_1 \xrightarrow{\mathcal{R}} t_2$ and $\text{base}(u_1) = t_1$ then $u_1 \xrightarrow{\mathcal{S}} u_2$ with $\text{base}(u_2) = t_2$

Lemma

enrichment \mathcal{S} of \mathcal{R} is terminating $\implies \mathcal{R}$ is terminating

Definition

TRS \mathcal{R} over signature \mathcal{F} is **locally terminating** if every restriction of \mathcal{R} to finite signature $\mathcal{G} \subseteq \mathcal{F}$ is terminating

Definitions

TRS \mathcal{R} over signature \mathcal{F} $L \subseteq \mathcal{T}(\mathcal{F})$

- $(\xrightarrow{\mathcal{R}}^*)(L) = \{u \in \mathcal{T}(\mathcal{F}) \mid t \xrightarrow{\mathcal{R}}^* u \text{ for some } t \in L\}$
- \mathcal{R} is **compact** for L if $(\xrightarrow{\mathcal{R}}^*)(L) \subseteq \mathcal{T}(\mathcal{G})$ for finite signature $\mathcal{G} \subseteq \mathcal{F}$

Lemma

\mathcal{R} is locally terminating and compact for $\mathcal{T}(\mathcal{F}) \implies \mathcal{R}$ is terminating

Idea

prove termination of TRS \mathcal{R} by constructing enrichment \mathcal{S} such that

- 1 \mathcal{S} is locally terminating
- 2 \mathcal{S} is compact for $\mathcal{T}(\mathcal{F})$

Definition (TRS \mathcal{R} over signature \mathcal{F})

- signature $\mathcal{F}_{\mathbb{N}} = \{f_n \mid f \in \mathcal{F} \text{ and } n \in \mathbb{N}\}$
- mappings $\text{base}: \mathcal{F}_{\mathbb{N}} \rightarrow \mathcal{F}$ $\text{height}: \mathcal{F}_{\mathbb{N}} \rightarrow \mathbb{N}$ $\text{lift}_c: \mathcal{F} \rightarrow \mathcal{F}_{\mathbb{N}}$ for $c \in \mathbb{N}$

$$\text{base}(f_n) = f \quad \text{height}(f_n) = n \quad \text{lift}_c(f) = f_c$$

- TRS $\text{match}(\mathcal{R})$ consists of all rules $l \rightarrow \text{lift}_c(r)$ such that
 - 1 $\text{base}(l) \rightarrow r \in \mathcal{R}$
 - 2 $c = 1 + \min\{\text{height}(l(p)) \mid p \in \mathcal{FPos}(l)\}$

Example

$$\begin{array}{ll} \mathcal{R} & f(g(a, x), y) \rightarrow h(f(x, y)) & f_0(g_0(a_0, x), y) \rightarrow h_1(f_1(x, y)) & \text{match}(\mathcal{R}) \\ & & f_0(g_0(a_1, x), y) \rightarrow h_1(f_1(x, y)) \\ & & f_2(g_1(a_1, x), y) \rightarrow h_2(f_2(x, y)) \\ & & f_7(g_4(a_3, x), y) \rightarrow h_4(f_4(x, y)) \\ & & \dots \end{array}$$

Lemma

$\text{match}(\mathcal{R})$ is enrichment of \mathcal{R} for every left-linear TRS \mathcal{R}

Definition

TRS \mathcal{R} over signature \mathcal{F} is **match-bounded** if

$$\left(\frac{*}{\text{match}(\mathcal{R})}\right)(\text{lift}_0(\mathcal{T}(\mathcal{F}))) \subseteq \mathcal{T}(\mathcal{F}_{\{0, \dots, c\}})$$

for some $c \in \mathbb{N}$

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 - Model Version
 - Quasi-Model Version
 - Root-Labeling

- Further Reading

Question

how to prove match-boundedness ?

Answer

use automata techniques

Idea

SRS \mathcal{R} over signature \mathcal{F}

- 1 construct finite automaton \mathcal{A} that accepts $\text{lift}_0(\mathcal{T}(\mathcal{F}))$
- 2 close \mathcal{A} under rewriting with respect to $\text{match}(\mathcal{R})$

Theorem

$\text{match}(\mathcal{R})$ is *locally terminating* for every SRS \mathcal{R}

Corollary

match-bounded SRSs are terminating

Remark

$\text{match}(\mathcal{R})$ is *not* locally terminating for arbitrary TRSs \mathcal{R}

Example

$\mathcal{R} \quad f(a, x) \rightarrow f(x, x) \quad f_1(a_0, x) \rightarrow f_1(x, x) \quad \dots \quad \text{match}(\mathcal{R})$

Definition

finite automaton $\mathcal{A} = (Q, \Sigma, \delta, s, F)$ is *compatible* with SRS \mathcal{R} over signature \mathcal{F} if

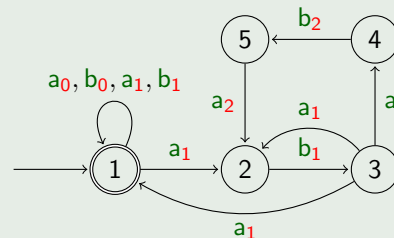
- 1 $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{L}(\mathcal{A})$
- 2 $\forall l \rightarrow r \in \mathcal{R} \quad \forall p, q \in Q \quad \text{if } p \xrightarrow{l} q \text{ then } p \xrightarrow{r} q$

Example

rewrite rule $a(a(x)) \rightarrow a(b(a(x)))$

match-bounded by 2

finite automaton



compatibility violations

- | | | | |
|-------------------------|-------------------------|-------------------------|-------------------------|
| $1 \xrightarrow{a_0} 1$ | $1 \xrightarrow{a_0} 1$ | $1 \xrightarrow{a_1} 2$ | $3 \xrightarrow{a_1} 1$ |
| $1 \xrightarrow{a_0} 1$ | $1 \xrightarrow{a_1} 2$ | $1 \xrightarrow{a_1} 2$ | $3 \xrightarrow{a_1} 2$ |
| $3 \xrightarrow{a_1} 1$ | $1 \xrightarrow{a_0} 1$ | $3 \xrightarrow{a_1} 1$ | $1 \xrightarrow{a_1} 1$ |
| $3 \xrightarrow{a_1} 1$ | $1 \xrightarrow{a_1} 2$ | $3 \xrightarrow{a_2} 4$ | $5 \xrightarrow{a_2} 2$ |

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Example

$$\mathcal{R} \quad f(g(x, y), a) \rightarrow h(f(x, y)) \quad \text{roof}(\mathcal{R})$$

$$f_0(g_0(x, y), a_0) \rightarrow h_1(f_1(x, y))$$

$$f_0(g_0(x, y), a_1) \rightarrow h_1(f_1(x, y))$$

$$f_2(g_1(x, y), a_1) \rightarrow h_3(f_3(x, y))$$

$$f_7(g_4(x, y), a_3) \rightarrow h_5(f_5(x, y))$$

$$\dots$$

Lemma

- $\text{top}(\mathcal{R})$ and $\text{roof}(\mathcal{R})$ are locally terminating
- $\text{top}(\mathcal{R})$ and $\text{roof}(\mathcal{R})$ are enrichments of *left-linear* \mathcal{R}

Lemma

$\text{match}(\mathcal{R})$ is locally terminating for *right-linear* TRSs

Corollary

linear match-bounded TRSs are terminating

Definitions

- TRS $\text{top}(\mathcal{R})$ consists of all rules $l \rightarrow \text{lift}_c(r)$ such that
 - 1 $\text{base}(l) \rightarrow r \in \mathcal{R}$
 - 2 $c = 1 + \text{height}(l(\epsilon))$
- TRS $\text{roof}(\mathcal{R})$ consists of all rules $l \rightarrow \text{lift}_c(r)$ such that
 - 1 $\text{base}(l) \rightarrow r \in \mathcal{R}$
 - 2 $c = 1 + \min\{\text{height}(l(p)) \mid p \in \mathcal{R}\text{Pos}_{\text{var}(r)}(l)\}$
- $\mathcal{R}\text{Pos}_V(t) = \{p \in \mathcal{F}\text{Pos}(t) \mid V \subseteq \text{Var}(t|_p)\}$

Definition

- TRS \mathcal{R} over signature \mathcal{F} is *top-bounded* if

$$\left(\frac{*}{\text{top}(\mathcal{R})}\right)(\text{lift}_0(\mathcal{T}(\mathcal{F}))) \subseteq \mathcal{T}(\mathcal{F}_{\{0, \dots, c\}})$$

for some $c \in \mathbb{N}$

- TRS \mathcal{R} over signature \mathcal{F} is *roof-bounded* if

$$\left(\frac{*}{\text{roof}(\mathcal{R})}\right)(\text{lift}_0(\mathcal{T}(\mathcal{F}))) \subseteq \mathcal{T}(\mathcal{F}_{\{0, \dots, c\}})$$

for some $c \in \mathbb{N}$

Corollary

left-linear top-bounded and roof-bounded TRSs are terminating

Lemma

top-boundedness \implies roof-boundedness \implies match-boundedness

Example

rewrite rule $f(f(x, a), a) \rightarrow f(x, f(x, a))$ roof-bounded by 1

not top-bounded

- $t_n = \begin{cases} a & \text{if } n = 0 \\ f(t_{n-1}, a) & \text{if } n > 0 \end{cases}$
- $\forall n \geq 0 \quad t_{n+2} = f(f(t_n, a), a) \rightarrow^{n+1} f(t_n, f(t_{n-1}, \dots, f(a, a) \dots))$

tree automaton

a_0	\rightarrow	0		
$f_0(0, 0)$	\rightarrow	0	compatibility violations	
a_1	\rightarrow	1	$f_0(f_0(0, a_0), a_0) \rightarrow^* 0$	$f_1(0, f_1(0, a_1)) \rightarrow^* 0$
$f_1(0, 1)$	\rightarrow	2	$f_1(f_0(0, a_0), a_1) \rightarrow^* 2$	$f_1(0, f_1(0, a_1)) \rightarrow^* 2$
$f_1(0, 2)$	\rightarrow	0		
$f_1(0, 2)$	\rightarrow	2		

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Definition

RFC (\mathcal{R}) is least extension of $\text{rhs}(\mathcal{R})$ closed under **narrowing**:

$t[r]_p \sigma \in \text{RFC}(\mathcal{R})$ whenever $t \in \text{RFC}(\mathcal{R})$ and $\exists p \in \mathcal{FPos}(t) \exists l \rightarrow r \in \mathcal{R}$ such that σ is mgu of $t|_p$ and l

Theorem

right-linear TRS \mathcal{R} is terminating $\iff \mathcal{R}$ is terminating on $\text{RFC}(\mathcal{R})$

Definition

TRS \mathcal{R} over signature \mathcal{F} is **RFC-match-bounded** if

$$\left(\frac{*}{\text{match}(\mathcal{R})} \right) (\text{lift}_0(\text{RFC}(\mathcal{R})\sigma_{\#})) \subseteq \mathcal{T}(\mathcal{F}_{\{0, \dots, c\}})$$

for some $c \in \mathbb{N}$

Notation

$\sigma_{\#}$ maps all variables to fresh constant $\#$

Definition

$\mathcal{R}_{\#}$ is least extension of \mathcal{R} such that $l[\#]_p \rightarrow r \sigma \in \mathcal{R}_{\#}$ whenever $\exists l \rightarrow r \in \mathcal{R}$ $\exists p \in \mathcal{FPos}(l) \setminus \{\epsilon\}$ with

$$\sigma(x) = \begin{cases} \# & \text{if } x \in \text{Var}(l|_p) \\ x & \text{otherwise} \end{cases}$$

Theorem

$\text{RFC}(\mathcal{R})\sigma_{\#} = \left(\frac{*}{\mathcal{R}_{\#}} \right) (\text{rhs}(\mathcal{R})\sigma_{\#})$ for linear TRSs \mathcal{R}

Example

$$\mathcal{R} \quad f(f(x, a), a) \rightarrow f(x, f(x, a)) \quad \mathcal{R}_\#$$

$$\begin{aligned} f(f(x, a), a) &\rightarrow f(x, f(x, a)) \\ f(f(x, \#), a) &\rightarrow f(x, f(x, a)) \\ f(f(x, a), \#) &\rightarrow f(x, f(x, a)) \\ f(f(x, \#), \#) &\rightarrow f(x, f(x, a)) \\ f(\#, a) &\rightarrow f(\#, f(\#, a)) \\ f(\#, \#) &\rightarrow f(\#, f(\#, a)) \end{aligned}$$

Extensions

- efficient exact construction of $(\xrightarrow[\text{match}(\mathcal{R})]{*})(\text{lift}_0(\mathcal{T}(\mathcal{F})))$ for SRSs
- extension to non-left-linear TRSs using quasi-deterministic automata
- DP processors based on bounds

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Definitions (TRS \mathcal{R} over signature \mathcal{F})

- **semantics**
 - \mathcal{F} -algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$
- **labeling**
 - sets of labels $L_f \subseteq A$ for every $f \in \mathcal{F}$
 - labeling functions $\ell_f: A^n \rightarrow L_f$ for every n -ary $f \in \mathcal{F}$ with $L_f \neq \emptyset$
- labeling of terms (for every assignment $\alpha: \mathcal{V} \rightarrow A$)

$$\text{lab}_\alpha(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f(\text{lab}_\alpha(t_1), \dots, \text{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f = \emptyset \\ f_a(\text{lab}_\alpha(t_1), \dots, \text{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f \neq \emptyset \end{cases}$$

with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \dots, [\alpha]_{\mathcal{A}}(t_n))$

- labeled TRS

$$\mathcal{R}_{\text{lab}} = \{ \text{lab}_\alpha(l) \rightarrow \text{lab}_\alpha(r) \mid l \rightarrow r \in \mathcal{R} \text{ and } \alpha: \mathcal{V} \rightarrow A \}$$

Example

- TRS \mathcal{R}

$$f(a, b, x) \rightarrow f(x, x, x)$$

- algebra \mathcal{A} is **model** of \mathcal{R}

$$A = \{0, 1\} \quad a_{\mathcal{A}} = 0 \quad b_{\mathcal{A}} = 1 \quad f_{\mathcal{A}}(x, y, z) = 0$$

- labeling ℓ

$$L_a = L_b = \emptyset \quad L_f = A \quad \ell_f(x, y, z) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

- TRS \mathcal{R}_{lab}

$$f_1(a, b, x) \rightarrow f_0(x, x, x)$$

is terminating because it lacks dependency pairs

Theorem

TRS \mathcal{R} over signature \mathcal{F} is terminating if

\exists algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$

\exists labeling ℓ

such that

1 $\mathcal{R} \subseteq =_{\mathcal{A}}$ “ \mathcal{A} is model of \mathcal{R} ”

2 \mathcal{R}_{lab} is terminating

Theorem (rephrased)

TRS \mathcal{R} , model \mathcal{A} , labeling ℓ (semantic labeling is *sound* and *complete*)

\mathcal{R} is terminating $\iff \mathcal{R}_{lab}$ is terminating

Example

- TRS \mathcal{R}

$$f(a, b, x) \rightarrow f(x, x, x) \quad a \rightarrow c \quad b \rightarrow c \quad f(x, y, z) \rightarrow c$$

- weakly monotone algebra \mathcal{A}

$$A = \{0, 1, 2\} \quad 2 > 0 \quad 1 > 0 \quad a_{\mathcal{A}} = 1 \quad b_{\mathcal{A}} = 2 \quad c_{\mathcal{A}} = f_{\mathcal{A}}(x, y, z) = 0$$

- weakly monotone labeling ℓ

$$L_a = L_b = L_c = \emptyset \quad L_f = \{0, 1\} \quad \ell_f(x, y, z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{if otherwise} \end{cases}$$

- TRS $\mathcal{R}_{lab} \cup Dec$

$$\begin{array}{llll} f_1(a, b, x) \rightarrow f_0(x, x, x) & a \rightarrow c & b \rightarrow c & f_0(x, y, z) \rightarrow c \\ f_1(x, y, z) \rightarrow f_0(x, y, z) & & & f_1(x, y, z) \rightarrow c \end{array}$$

is terminating because dependency graph lacks SCCs

Definitions

- weakly monotone \mathcal{F} -algebra $(\mathcal{A}, >, \geq)$ with $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ and $\geq = >^=$
- weakly monotone labeling functions $\ell_f: A^n \rightarrow L_f$ for every n -ary $f \in \mathcal{F}$ with $L_f \neq \emptyset$
- $Dec = \{f_a(x_1, \dots, x_n) \rightarrow f_b(x_1, \dots, x_n) \mid a, b \in L_f \text{ with } a > b\}$

Theorem

TRS \mathcal{R} over signature \mathcal{F} is terminating if

\exists weakly monotone algebra $(\mathcal{A}, >, \geq)$

\exists weakly monotone labeling ℓ

such that

1 $\mathcal{R} \subseteq \geq_{\mathcal{A}}$ “ \mathcal{A} is quasi-model of \mathcal{R} ”

2 $\mathcal{R}_{lab} \cup Dec$ is terminating

Example

- TRS \mathcal{R}

$$f(a, b, x) \rightarrow f(x, x, x) \quad a \rightarrow c \quad b \rightarrow c \quad f(x, y, z) \rightarrow c$$

- weakly monotone algebra \mathcal{A}

$$A = \{0, 1, 2\} \quad 2 > 0 \quad 1 > 0 \quad a_{\mathcal{A}} = 1 \quad b_{\mathcal{A}} = 2 \quad c_{\mathcal{A}} = f_{\mathcal{A}}(x, y, z) = 0$$

- weakly monotone labeling ℓ

$$L_a = L_b = L_c = \emptyset \quad L_f = \{0, 1\} \quad \ell_f(x, y, z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{if otherwise} \end{cases}$$

- TRS $\mathcal{R}_{lab} \cup Dec$

$$\begin{array}{llll} f_1(a, b, x) \rightarrow f_0(x, x, x) & a \rightarrow c & b \rightarrow c & f_0(x, y, z) \rightarrow c \\ f_1(x, y, z) \rightarrow f_0(x, y, z) & & & f_1(x, y, z) \rightarrow c \end{array}$$

is terminating because dependency graph lacks SCCs

Theorem (rephrased)

TRS \mathcal{R} , quasi-model \mathcal{A} , weakly monotone labeling ℓ

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{lab} \text{ is terminating}$$

Example

- TRS \mathcal{R}

$$\begin{array}{lll} \lambda(x) \circ y \rightarrow \lambda(x \circ (1 * (y \circ \uparrow))) & \text{id} \circ x \rightarrow x & 1 \circ (x * y) \rightarrow x \\ (x * y) \circ z \rightarrow (x \circ z) * (y \circ z) & 1 \circ \text{id} \rightarrow 1 & \uparrow \circ (x * y) \rightarrow y \\ (x \circ y) \circ z \rightarrow x \circ (y \circ z) & \uparrow \circ \text{id} \rightarrow \uparrow & \end{array}$$

- weakly monotone algebra \mathcal{A}
 - carrier $A = \mathbb{N}$ with standard order $>$
 - interpretations

$$\begin{array}{ll} \lambda_{\mathcal{A}}(x) = x + 1 & \circ_{\mathcal{A}}(x, y) = x + y \\ *_{\mathcal{A}}(x, y) = \max(x, y) & 1_{\mathcal{A}} = \uparrow_{\mathcal{A}} = \text{id}_{\mathcal{A}} = 0 \end{array}$$

Example (cont'd)

- weakly monotone labeling ℓ

$$L_{\lambda} = L_{*} = L_1 = L_{\uparrow} = \emptyset \quad L_{\circ} = \mathbb{N} \quad \ell_{\circ}(x, y) = x + y$$

- TRS $\mathcal{R}_{lab} \cup \mathcal{D}ec \quad \forall i, j, k \geq 0 \quad \forall l > m \geq 0$

$$\begin{array}{l} \lambda(x) \circ_{i+j+1} y \rightarrow \lambda(x \circ_{i+j} (1 * (y \circ_j \uparrow))) \\ (x * y) \circ_{\max(i,j)+k} z \rightarrow (x \circ_{i+k} z) * (y \circ_{j+k} z) \\ (x \circ_{i+j} y) \circ_{i+j+k} z \rightarrow x \circ_{i+j+k} (y \circ_{j+k} z) \\ \text{id} \circ_i x \rightarrow x \quad 1 \circ_{\max(i,j)} (x * y) \rightarrow x \\ 1 \circ_0 \text{id} \rightarrow 1 \quad \uparrow \circ_{\max(i,j)} (x * y) \rightarrow y \\ \uparrow \circ_0 \text{id} \rightarrow \uparrow \quad x \circ_l y \rightarrow x \circ_m y \end{array}$$

- LPO with precedence

$$\dots > \circ_2 > \circ_1 > \circ_0 > *, \lambda, 1, \uparrow$$

Extensions

- quasi-model condition required for usable rules only (predictive labeling)
- incorporation of argument filters
- DP processors based on semantic labeling

Questions

- how to choose (quasi-)model ?
- how to choose labeling ?

Answers

- ...
- root-labeling

Definition

- algebra $\mathcal{A}_{\mathcal{F}}$
 - $A_{\mathcal{F}} = \mathcal{F}$
 - $f_{\mathcal{A}_{\mathcal{F}}}(x_1, \dots, x_n) = f \quad \forall n\text{-ary } f \in \mathcal{F}$
- labeling
 - $L_f = \mathcal{F}^n$
 - $\ell_f(x_1, \dots, x_n) = (x_1, \dots, x_n)$

Example

$$\mathcal{R} \quad \begin{array}{lll} a(b(x)) \rightarrow c(x) & a_b(b_a(x)) \rightarrow c_a(x) & \mathcal{R}_{rlab} \\ & a_b(b_b(x)) \rightarrow c_b(x) & \\ & a_b(b_c(x)) \rightarrow c_c(x) & \end{array}$$

Problem

$$\text{no model: } \forall \alpha \quad [\alpha](a(b(x))) = a \neq c = [\alpha](c(x))$$

Definitions (TRS \mathcal{R} over signature \mathcal{F})

- **root-preserving** rules
 $\mathcal{R}_p = \{ l \rightarrow r \in \mathcal{R} \mid \text{root}(l) = \text{root}(r) \}$
- **root-altering** rules
 $\mathcal{R}_a = \{ l \rightarrow r \in \mathcal{R} \mid \text{root}(l) \neq \text{root}(r) \}$
- **flat** contexts
 $\mathcal{FC} = \{ g(x_1, \dots, x_{i-1}, \square, x_i, \dots, x_n) \mid g \in \mathcal{F} \text{ has arity } n \text{ and } 1 \leq i \leq n \}$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}_p \cup \{ C[l] \rightarrow C[r] \mid l \rightarrow r \in \mathcal{R}_a \text{ and } C \in \mathcal{FC} \}$

Lemma

- \mathcal{R} is terminating if and only if $\mathcal{FC}(\mathcal{R})$ is terminating
- $\mathcal{A}_{\mathcal{F}}$ is model of $\mathcal{FC}(\mathcal{R})$

Example

\mathcal{R}	$a(b(x)) \rightarrow c(x)$	
$\mathcal{FC}(\mathcal{R})$	$a(a(b(x))) \rightarrow a(c(x))$ $b(a(b(x))) \rightarrow b(c(x))$ $c(a(b(x))) \rightarrow c(c(x))$	$a(\square)$ $b(\square)$ $c(\square)$
$\mathcal{FC}(\mathcal{R})_{\text{rlab}}$	$a_a(a_b(b_a(x))) \rightarrow a_c(c_a(x))$ $a_a(a_b(b_b(x))) \rightarrow a_c(c_b(x))$ $a_a(a_b(b_c(x))) \rightarrow a_c(c_c(x))$ $b_a(a_b(b_a(x))) \rightarrow b_c(c_a(x))$ $b_a(a_b(b_b(x))) \rightarrow b_c(c_b(x))$ $b_a(a_b(b_c(x))) \rightarrow b_c(c_c(x))$ $c_a(a_b(b_a(x))) \rightarrow c_c(c_a(x))$ $c_a(a_b(b_b(x))) \rightarrow c_c(c_b(x))$ $c_a(a_b(b_c(x))) \rightarrow c_c(c_c(x))$	

Corollary

\mathcal{R} is terminating if and only if $\mathcal{FC}(\mathcal{R})_{\text{rlab}}$ is terminating

Example (1)






- \mathcal{R} $f(a, b, x) \rightarrow f(x, x, x)$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$
- $\mathcal{FC}(\mathcal{R})_{\text{rlab}}$ $f_{(a,b,a)}(a, b, x) \rightarrow f_{(a,a,a)}(x, x, x)$
 $f_{(a,b,b)}(a, b, x) \rightarrow f_{(b,b,b)}(x, x, x)$
 $f_{(a,b,f)}(a, b, x) \rightarrow f_{(f,f,f)}(x, x, x)$
- $\mathcal{FC}(\mathcal{R})_{\text{rlab}}$ is terminating because it lacks dependency pairs

Example (2)

- \mathcal{R} $f(f(x)) \rightarrow f(g(f(x)))$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$
- $\mathcal{FC}(\mathcal{R})_{\text{rlab}}$ $f_f(f_f(x)) \rightarrow f_g(g_f(f_f(x)))$
 $f_f(f_g(x)) \rightarrow f_g(g_f(f_g(x)))$
- $\mathcal{FC}(\mathcal{R})_{\text{rlab}}$ is polynomially terminating over \mathbb{N}

Outline

- Match-Bounds
- Semantic Labeling
- **Further Reading**

-  [On Tree Automata that Certify Termination of Left-Linear Term Rewriting Systems](#)
Alfons Geser, Dieter Hofbauer, Johannes Waldmann and Hans Zantema
I&C 205(4), pp. 512 – 534, 2007
-  [Match-Bounds Revisited](#)
Martin Korp and Aart Middeldorp
I&C, 2009, to appear
-  [Termination of Term Rewriting by Semantic Labelling](#)
Hans Zantema
FI 24(1,2), pp. 89 – 105, 1995
-  [Innermost Termination of Rewrite Systems by Labeling](#)
René Thiemann and Aart Middeldorp
Proc. 7th WRS, ENTCS 204, pp. 3 – 19, 2008
-  [Root-Labeling](#)
Christian Sternagel and Aart Middeldorp
Proc. 19th RTA, LNCS 5117, pp. 336 – 350, 2008