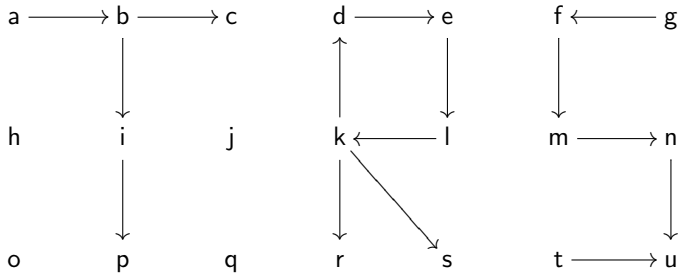
Exercises

1. Study the slides.
2. Consider the term $t = (\mathsf{s}(0) + x) + \mathsf{s}(\mathsf{s}(0))$.
   (a) Determine $\mathcal{V}\mathsf{ar}(t)$, $\mathcal{F}\mathsf{un}(t)$, $\mathsf{root}(t)$, $|t|$, and $\|t\|$.
   (b) Determine all subterms of t and their positions.
3. Let $t = x + (y + (x + y))$. Determine $t\sigma$ and $\mathcal{D}\mathsf{om}(\sigma)$ for the following substitutions σ.
   (a) $\{x \mapsto y\}$
   (b) $\{x \mapsto y + x,\ y \mapsto y + y,\ z \mapsto x\}$
   (c) $\{x \mapsto 0 + z,\ y \mapsto \mathsf{s}(0),\ z \mapsto x + x\}$
4. Which of the following terms match the term $x + (y + (x + y))$?
   (a) $x + (y + z)$
   (b) $(x + y) + z$
   (c) $x + x$
   (d) x
5. Consider the following ARS:



   (a) Complete the following table:

	SN	WN	UN	CR	WCR
a				×	
d					
f					
h		✓			
k					

   (b) Suppose we add the arrows $\mathsf{c} \to \mathsf{d}$ and $\mathsf{s} \to \mathsf{t}$.
      i. Does $\mathsf{a} \downarrow \mathsf{g}$ hold?
      ii. Does $\mathsf{a} \uparrow \mathsf{g}$ hold?
6. Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS and let a be an arbitrary element of A.
   (a) Which of the following implications hold?
      i. $\mathrm{SN}(a) \Rightarrow \mathrm{WN}(a)$
      ii. $\mathrm{CR}(a) \Rightarrow \mathrm{WCR}(a)$
      iii. $\mathrm{CR}(a) \Rightarrow \mathrm{UN}(a)$

(b) Show that the implication $\mathrm{WN}(a)$ & $\mathrm{UN}(a) \Rightarrow \mathrm{CR}(a)$ does not hold in general.

(c) Does the implication $\mathrm{WN}(a)$ & $\mathrm{UN}(\mathcal{A}) \Rightarrow \mathrm{CR}(a)$ hold?

7. An ARS $\mathcal{A} = \langle A, \to \rangle$ has the *normal form property* (NFP) if all elements that have a normal form are confluent ($\forall a \in A$ if $\mathrm{WN}(a)$ then $\mathrm{CR}(a)$).

   (a) Show that every confluent ARS has the normal form property.

   (b) Does the converse hold?

   (c) Show that the following statements are equivalent:

      i. $\mathcal{A}$ has the normal form property,

      ii. $\leftarrow \cdot \to^! \subseteq \to^!$,

      iii. every element convertible to a normal form rewrites to that normal form.

8. (a) Order the multisets

   - $M_1 = \{2\}$,
   - $M_2 = \{1, 3\}$,
   - $M_3 = \{1, 1, 1, 2\}$,
   - $M_4 = \{2, 2, 2, 2\}$, and
   - $M_5 = \{1, 2\}$

   with respect to the multiset extension of the standard order on natural numbers.

   (b) Order the multisets

   - $N_1 = \{\mathsf{0}, x, \mathsf{0}, y, \mathsf{0}, \mathsf{0}\}$,
   - $N_2 = \{\mathsf{s}(x), \mathsf{s}(\mathsf{0}), \mathsf{0}, \mathsf{s}(\mathsf{0}), y \times \mathsf{s}(\mathsf{0}), \mathsf{s}(\mathsf{s}(\mathsf{0}))\}$,
   - $N_3 = \{\mathsf{s}(\mathsf{0}), y, \mathsf{s}(x)\}$,
   - $N_4 = \{\mathsf{s}(y \times \mathsf{s}(\mathsf{0})), \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0}))), \mathsf{s}(x)\}$, and
   - $N_5 = \{\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0}))), \mathsf{s}(y \times \mathsf{s}(\mathsf{0})) + \mathsf{s}(x)\}$

   with respect to the multiset extension of $\rhd$.

9. Consider the ES $\mathcal{E}$ consisting of the three equations

   $$\mathsf{f}(x) \approx x \qquad \mathsf{f}(\mathsf{f}(\mathsf{a})) \approx \mathsf{g}(x, x) \qquad \mathsf{g}(x, \mathsf{f}(x)) \approx \mathsf{b}$$

   Which of the following equations belong to the equational theory of $\mathcal{E}$?

   (a) $\mathsf{a} \approx \mathsf{b}$

   (b) $\mathsf{g}(x, y) \approx \mathsf{g}(y, x)$

   (c) $\mathsf{g}(\mathsf{f}(\mathsf{a}), \mathsf{a}) \approx \mathsf{f}(\mathsf{b})$

10. Which of the following equations are valid in group theory?

    (a) $(x \cdot (y^- \cdot x)^-) \cdot y \approx \mathsf{e}$

    (b) $(x \cdot x^-) \cdot ((y^- \cdot (\mathsf{e}^- \cdot x))^- \cdot y^-) \approx (x^- \cdot \mathsf{e})^-$

    (c) $(x^{--} \cdot (x \cdot (x \cdot \mathsf{e})^-))^- \approx x^{-----}$

11. Consider the term $t = (\mathsf{0} + \mathsf{s}(\mathsf{0})) + (\mathsf{s}(\mathsf{s}(\mathsf{0})) + (\mathsf{0} + \mathsf{0}))$. Which terms are denoted by the following expressions?

    (a) $t|_{21}$

    (b) $t[\mathsf{0} + \mathsf{s}(\mathsf{0})]_{121}$

    (c) $(t|_2[t|_1[t|_{22}]_{21}]_{11})|_1[t|_{211}[t|_{121}]_1]_{12}$

⋆12. Prove that $\mathsf{TAGCTAGCTAGCT} \approx_{\mathcal{E}} \mathsf{CTGACTGACT}$ with respect to the ES $\mathcal{E}$ consisting of the following equations between strings:

$$\mathsf{TCAT} \approx \mathsf{T} \qquad \mathsf{GAG} \approx \mathsf{AG} \qquad \mathsf{CTC} \approx \mathsf{TC} \qquad \mathsf{AGTA} \approx \mathsf{A} \qquad \mathsf{TAT} \approx \mathsf{CT}$$

13. Is the TRS consisting of the rewrite rules

$$\mathsf{double}(\mathsf{0}) \to \mathsf{0}$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

terminating?