Solutions

2. (a) $\mathcal{V}\mathsf{ar}(t) = \{x\}$, $\mathcal{F}\mathsf{un}(t) = \{\mathsf{s}, \mathsf{0}, +\}$, $|t| = 8$, $\|t\| = 7$
   (b) t at position ϵ, $\mathsf{s}(\mathsf{0}) + x$ at position 1, $\mathsf{s}(\mathsf{0})$ at positions 11 and 21, $\mathsf{0}$ at positions 111 and 211, x at position 12, $\mathsf{s}(\mathsf{s}(\mathsf{0}))$ at position 2

3. (a) $t\sigma = y + (y + (y + y))$, $\mathcal{D}\mathsf{om}(\sigma) = \{x\}$
   (b) $t\sigma = (y + x) + ((y + y) + ((y + x) + (y + y)))$, $\mathcal{D}\mathsf{om}(\sigma) = \{x, y, z\}$
   (c) $t\sigma = (\mathsf{0} + z) + (\mathsf{s}(\mathsf{0}) + ((\mathsf{0} + z) + \mathsf{s}(\mathsf{0})))$, $\mathcal{D}\mathsf{om}(\sigma) = \{x, y, z\}$

4. The terms $x + (y + z)$ and x.

5. (a)

	SN	WN	UN	CR	WCR
a	✓	✓	×	×	✓
d	×	✓	×	×	✓
f	✓	✓	✓	✓	✓
h	✓	✓	✓	✓	✓
k	×	✓	×	×	×

   (b) i. Yes: $\mathsf{a} \to \mathsf{b} \to \mathsf{c} \to \mathsf{d} \to \mathsf{e} \to \mathsf{l} \to \mathsf{k} \to \mathsf{s} \to \mathsf{t} \to \mathsf{u} \leftarrow \mathsf{n} \leftarrow \mathsf{m} \leftarrow \mathsf{f} \leftarrow \mathsf{g}$
   ii. No. The only element that rewrites to a is a itself and the only element that rewrites to g is g itself.

6. (a) All three implications are valid.
   (b) The ARS

$$\mathsf{a} \longleftarrow \mathsf{b} \longrightarrow \mathsf{c} \circlearrowleft$$

   constitutes a counterexample. Element b is weakly normalizing and has unique normal forms but it is not confluent.
   (c) No. The ARS of part (b) is a counterexample. (The implication $\mathrm{WN}(\mathcal{A})$ & $\mathrm{UN}(a) \Rightarrow \mathrm{CR}(a)$ does hold in general.)

7. (a) Obvious.
   (b) No. The ARS

$$\circlearrowright \mathsf{a} \longleftarrow \mathsf{b} \longrightarrow \mathsf{c} \circlearrowleft$$

   has no normal forms, so the normal form property is vacuously satisfied, but it is not confluent.
   (c) We have to show the equivalence of (1) the normal form property, (2) $\leftarrow \cdot \to^! \subseteq \to^!$, and (3) every element convertible to a normal form rewrites to that normal form.

(1) ⇒ (2) Suppose $b \leftarrow a \to^! c$. Since a has a normal form, a is confluent according to the normal form property. So $b \downarrow c$, which is only possible if $b \to^! c$.

(2) ⇒ (3) Suppose that $\leftarrow \cdot \to^! \subseteq \to^!$ and let $a \leftrightarrow^* b$ with b a normal form. We show that $a \to^! b$ by induction on the length of the conversion between a and b. The case of zero length is trivial. Let $a \leftrightarrow a' \leftrightarrow^* b$. From the induction hypothesis we obtain $a' \to^! b$. If $a \to a'$ then clearly $a \to^! b$. Otherwise $a \leftarrow a'$ and hence we obtain $a \to^! b$ from the inclusion $\leftarrow \cdot \to^! \subseteq \to^!$.

(3) ⇒ (1) Suppose $a \to^! b$. We have to show that a is confluent. Let c be an arbitrary reduct of a. Clearly $c \leftrightarrow^* b$. By assumption $c \to^! b$. So every reduct of a rewrites to b. This implies confluence.

8. (a) $M_2 >_{\text{mul}} M_4 >_{\text{mul}} M_3 >_{\text{mul}} M_5 >_{\text{mul}} M_1$
   (b) $N_5 \rhd_{\text{mul}} N_4 \rhd_{\text{mul}} N_2 \rhd_{\text{mul}} N_3 \rhd_{\text{mul}} N_1$

9. (a) The following proof tree shows that $\mathsf{a} \approx_{\mathcal{E}} \mathsf{b}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{f(f(a))} \approx \mathsf{f(a)}}\,[\mathsf{a}] \quad \dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}]}{\mathsf{f(f(a))} \approx \mathsf{a}}\,[\mathsf{t}]}{\mathsf{a} \approx \mathsf{f(f(a))}}\,[\mathsf{s}] \quad \dfrac{}{\mathsf{f(f(a))} \approx \mathsf{g(a,a)}}\,[\mathsf{a}]}{\mathsf{a} \approx \mathsf{g(a,a)}}\,[\mathsf{t}] \quad \dfrac{\dfrac{\dfrac{}{\mathsf{a} \approx \mathsf{a}}\,[\mathsf{r}] \quad \dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}]}{\mathsf{g(a,f(a))} \approx \mathsf{g(a,a)}}\,[\mathsf{c}]}{\mathsf{g(a,a)} \approx \mathsf{g(a,f(a))}}\,[\mathsf{s}]}{\mathsf{a} \approx \mathsf{g(a,f(a))}}\,[\mathsf{t}] \quad \dfrac{}{\mathsf{g(a,f(a))} \approx \mathsf{b}}\,[\mathsf{a}]}{\mathsf{a} \approx \mathsf{b}}\,[\mathsf{t}]$$

Using Birkhoff's theorem, it follows that $\mathsf{a} \approx \mathsf{b}$ belongs to the equational theory of $\mathcal{E}$. You may find the following equational proof easier:

$$\mathsf{a} \leftarrow \mathsf{f(a)} \leftarrow \mathsf{f(f(a))} \rightarrow \mathsf{g(a,a)} \leftarrow \mathsf{g(a,f(a))} \rightarrow \mathsf{b}$$

   (b) Consider the algebra $\mathcal{A}$ with carrier $A = \{0,1\}$ and interpretations $\mathsf{a}_{\mathcal{A}} = \mathsf{b}_{\mathcal{A}} = 0$, $\mathsf{f}_{\mathcal{A}}(0) = 0$, $\mathsf{f}_{\mathcal{A}}(1) = 1$, $\mathsf{g}_{\mathcal{A}}(0,0) = \mathsf{g}_{\mathcal{A}}(0,1) = \mathsf{g}_{\mathcal{A}}(1,1) = 0$, and $\mathsf{g}_{\mathcal{A}}(1,0) = 1$. We have $\mathsf{f}(x) =_{\mathcal{A}} x$, $\mathsf{f(f(a))} =_{\mathcal{A}} \mathsf{g}(x,x)$, and $\mathsf{g}(x,\mathsf{f}(x)) =_{\mathcal{A}} \mathsf{b}$, so $\mathcal{A}$ is a model for $\mathcal{E}$. The equation $\mathsf{g}(x,y) \approx \mathsf{g}(y,x)$ is not valid in $\mathcal{A}$ because $\mathsf{g}_{\mathcal{A}}(0,1) = 0 \neq 1 = \mathsf{g}_{\mathcal{A}}(1,0)$. Hence $\mathsf{g}(x,y) \approx \mathsf{g}(y,x)$ does not belong to the equational theory of $\mathcal{E}$.
   (c) The following proof tree shows that $\mathsf{g(f(a),a)} \approx_{\mathcal{E}} \mathsf{f(b)}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}] \quad \dfrac{\dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}]}{\mathsf{a} \approx \mathsf{f(a)}}\,[\mathsf{s}]}{\mathsf{g(f(a),a)} \approx \mathsf{g(a,f(a))}}\,[\mathsf{c}] \quad \dfrac{}{\mathsf{g(a,f(a))} \approx \mathsf{b}}\,[\mathsf{a}]}{\mathsf{g(f(a),a)} \approx \mathsf{b}}\,[\mathsf{t}] \quad \dfrac{\dfrac{}{\mathsf{f(b)} \approx \mathsf{b}}\,[\mathsf{a}]}{\mathsf{b} \approx \mathsf{f(b)}}\,[\mathsf{s}]}{\mathsf{g(f(a),a)} \approx \mathsf{f(b)}}\,[\mathsf{t}]$$

Using Birkhoff's theorem, it follows that $\mathsf{g(f(a),a)} \approx \mathsf{f(b)}$ belongs to the equational theory of $\mathcal{E}$. Again, we present a simpler equational proof:

$$\mathsf{g(f(a),a)} \rightarrow \mathsf{g(a,a)} \leftarrow \mathsf{g(a,(f(a))} \rightarrow \mathsf{b} \leftarrow \mathsf{f(b)}$$

10. The idea is to compute the normal forms of both sides of the given equations with respect to the complete TRS on slide 23 of lecture 3.

   (a) $(x \cdot (y^- \cdot x)^-) \cdot y \approx \mathsf{e}$ We have $(x \cdot (y^- \cdot x)^-) \cdot y \rightarrow (x \cdot (x^- \cdot y^{--})) \cdot y \rightarrow y^{--} \cdot y \rightarrow y \cdot y$ and the normal form $y \cdot y$ is different from e. Hence the equation $(x \cdot (y^- \cdot x)^-) \cdot y \approx \mathsf{e}$ is not valid in group theory.
   (b) We have $(x \cdot x^-) \cdot ((y^- \cdot (\mathsf{e}^- \cdot x))^- \cdot y^-) \rightarrow^* \mathsf{e} \cdot ((y^- \cdot (\mathsf{e} \cdot x))^- \cdot y^-) \rightarrow^* (y^- \cdot x)^- \cdot y^- \rightarrow (x^- \cdot y^{--}) \cdot y^- \rightarrow (x^- \cdot y) \cdot y^- \rightarrow x^- \cdot (y \cdot y^-) \rightarrow x^- \cdot \mathsf{e} \rightarrow x^-$ and $(x^- \cdot \mathsf{e})^- \rightarrow x^{--} \rightarrow x$. Since x^- and x are different normal forms, the given equation is not valid in group theory.
   (c) Both sides rewrite to x^-: $(x^{--} \cdot (x \cdot (x \cdot \mathsf{e})^-))^- \rightarrow^* (x \cdot (x \cdot x^-))^- \rightarrow (x \cdot \mathsf{e})^- \rightarrow x^-$ and $x^{-----} \rightarrow x^{---} \rightarrow x^-$. Hence the given equation is valid in group theory.

11. Consider the term $t = (\mathsf{0} + \mathsf{s(0)}) + (\mathsf{s(s(0))} + (\mathsf{0} + \mathsf{0}))$. Which terms are denoted by the following expressions?

   (a) $t|_{21} = \mathsf{s(s(0))}$
   (b) $t[\mathsf{0} + \mathsf{s(0)}]_{121} = (\mathsf{0} + \mathsf{s}(\mathsf{0} + \mathsf{s(0)})) + (\mathsf{s(s(0))} + (\mathsf{0} + \mathsf{0}))$
   (c) $(t|_2[t|_1[t|_{22}]_{21}]_{11})|_1[t|_{211}[t|_{121}]_1]_{12} = \mathsf{s}(\mathsf{0} + \mathsf{s(0)})$

$\star$12. (This can be easily solved after lecture 6; a solution will be shown in lecture 7.)

13. Yes. Define

$$\phi(t) = \begin{cases} 0 & \text{if } t = \mathsf{0} \text{ or } t \in \mathcal{V} \\ \phi(u) + 1 & \text{if } t = \mathsf{s}(u) \\ 2\phi(u) & \text{if } t = \mathsf{double}(u) \end{cases}$$

and

$$\psi(t) = \sum \{\phi(u) \mid u \trianglelefteq t \text{ and } \mathsf{root}(u) = \mathsf{double}\}$$

It is easy to show that $\phi(t) = \phi(u)$ whenever $s \to t$. Using this fact, it can be shown that $\psi(t) > \psi(u)$ whenever $s \to t$. Since the standard order $>$ on natural numbers is well-founded, it follows that the TRS is terminating. (After lecture 4, proving termination is a piece of cake.)