Solutions

2. Consider the following interpretation in $\mathbb{N} \setminus \{0\}$: $\mathsf{f}_\mathbb{N}(x) = x + 1$ and $\otimes_\mathbb{N}(x, y) = x(y + 1)$. Since

$$\begin{aligned} (x+1)(y+2) &> x(y+1)+1 \\ (x+1)((y+1)(z+1)+1) &> (x(y+1)+1)(z+1) \\ x(y+1)(z+1) &> x(y(z+1)+1) \end{aligned}$$

for all $x, y, z \geqslant 1$, the TRS is polynomially terminating. If we change the interpretation of $\otimes$ to $\otimes_\mathbb{N}(x, y) = (x+1)(y+2)-1 = xy+2x+y+1$, the resulting inequalities are satisfied for all $x, y, z \in \mathbb{N}$.

⋆3. Yes. By taking the polynomial interpretation $\mathsf{0}_\mathbb{N} = 0$, $\mathsf{s}_\mathbb{N}(x) = x + 1$, and $\mathsf{f}_\mathbb{N}(x) = 2x^2 - x + 1$, the rewrite rules are transformed into the following constraints:

$$\begin{aligned} 1 &> 0 & \qquad 2 &> 1 \\ 2 &> 1 & \qquad 3 &> 2 \\ 7 &> 6 & \qquad 8 &> 7 \end{aligned}$$

Note that $\mathsf{f}_\mathbb{N}$ is well-defined (i.e., $\mathsf{f}_\mathbb{N}(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$) and strictly monotone.

4. (a) Suppose to the contrary that $\mathcal{R}$ is polynomially terminating. So there exists a compatible well-founded monotone algebra $(\mathbb{N}, \{\mathsf{a}_\mathbb{N}, \mathsf{b}_\mathbb{N}, \mathsf{f}_\mathbb{N}, \mathsf{g}_\mathbb{N}\}, >)$ with monotone functions $\mathsf{f}_\mathbb{N}$ and $\mathsf{g}_\mathbb{N}$ Consider $\mathsf{a}_\mathbb{N}$ and $\mathsf{b}_\mathbb{N}$. If $\mathsf{a}_\mathbb{N} > \mathsf{b}_\mathbb{N}$ then $\mathsf{g}_\mathbb{N}(\mathsf{a}_\mathbb{N}) > \mathsf{g}_\mathbb{N}(\mathsf{b}_\mathbb{N})$ by monotonicity of $\mathsf{g}_\mathbb{N}$. So the second rule is incompatible with the algebra. If $\mathsf{b}_\mathbb{N} > \mathsf{a}_\mathbb{N}$ then we obtain a contradiction with the first rule. Finally, if $\mathsf{a}_\mathbb{N} = \mathsf{b}_\mathbb{N}$ then both rules are incompatible with the algebra.

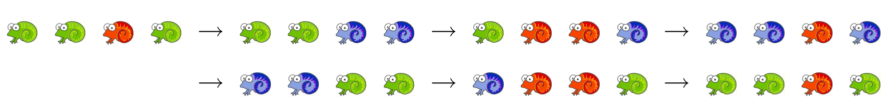
   (b) Termination of $\mathcal{R}$ can be shown by taking the well-founded monotone algebra $(\mathcal{A}, >)$ with carrier $A = \{a, b, \bot, \top\}$, well-founded order $\top > \bot$, and interpretations $\mathsf{a}_\mathcal{A} = a$, $\mathsf{b}_\mathcal{A} = b$, $\mathsf{f}_\mathcal{A}(a) = \mathsf{g}_\mathcal{A}(b) = \top$, $\mathsf{f}_\mathcal{A}(b) = \mathsf{g}_\mathcal{A}(a) = \bot$, $\mathsf{f}_\mathcal{A}(\bot) = \mathsf{g}_\mathcal{A}(\bot) = \bot$, and $\mathsf{f}_\mathcal{A}(\top) = \mathsf{g}_\mathcal{A}(\top) = \top$.

5. No. For the TRS of exercise 2, the first rule requires $\mathsf{f} > \otimes$ whereas the second rule requires $\otimes > \mathsf{f}$. No precedence satisfied both conditions. For the TRS of exercise 4, the first rule requires $\mathsf{a} > \mathsf{b}$ whereas the second rule requires $\mathsf{b} > \mathsf{a}$.

6. Suppose $s \rhd t$. So $s \trianglerighteq t$ and $s \neq t$. We use induction on the definition of $\trianglerighteq$. In the base case $s = t$ and there is nothing to prove. In the induction step $s = f(s_1, \ldots, s_n)$ and $s_i \trianglerighteq t$ for some $1 \leqslant i \leqslant n$. By definition, $s_i = t$ or $s_i \rhd t$. In the latter case we obtain $s_i >_{\mathsf{lpo}} t$ from the induction hypothesis. So in both cases we have $s >_{\mathsf{lpo}} t$ by the third clause in the definition of LPO.

7. Take the precedence $\mathsf{g} > \mathsf{f}$.

⋆8. No, because of the following cycle:



9. (a) The sequence

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x,y),x,y) \approx \mathsf{f}(z,\mathsf{g}(y,y),y) &\Rightarrow_{[\mathsf{d}]} && \mathsf{g}(x,y) \approx z, x \approx \mathsf{g}(y,y), y \approx y \\ &\Rightarrow_{[\mathsf{v}],\,\{z \mapsto \mathsf{g}(x,y)\}} && x \approx \mathsf{g}(y,y), y \approx y \\ &\Rightarrow_{[\mathsf{v}],\,\{x \mapsto \mathsf{g}(y,y)\}} && y \approx y \\ &\Rightarrow_{[\mathsf{t}]} && \square \end{aligned}$$

   yields the mgu $\{x \mapsto \mathsf{g}(y,y), z \mapsto \mathsf{g}(\mathsf{g}(y,y),y)\}$.

(b) The sequence

$$\begin{aligned}\mathsf{g}(\mathsf{h}(x), \mathsf{g}(x,y)) \approx \mathsf{g}(z, \mathsf{g}(\mathsf{g}(x,x), z)) &\Rightarrow_{[\mathsf{d}]} \mathsf{h}(x) \approx z, \mathsf{g}(x,y) \approx \mathsf{g}(\mathsf{g}(x,x), z)\\ &\Rightarrow_{[\mathsf{v}]} \mathsf{g}(x,y) \approx \mathsf{g}(\mathsf{g}(x,x), \mathsf{h}(x))\\ &\Rightarrow_{[\mathsf{d}]} x \approx \mathsf{g}(x,x), y \approx \mathsf{h}(x))\\ &\Rightarrow_{[\mathsf{v}]} x \approx \mathsf{g}(x,x)\end{aligned}$$

shows that $\mathsf{g}(\mathsf{h}(x), \mathsf{g}(x,y))$ and $\mathsf{g}(z, \mathsf{g}(\mathsf{g}(x,x), z))$ are not unifiable.

(c) Likewise, the terms $\mathsf{f}(x, \mathsf{g}(x,y), \mathsf{h}(y))$ and $\mathsf{f}(\mathsf{g}(z,z), x, x)$ are not unifiable:

$$\begin{aligned}\mathsf{f}(x, \mathsf{g}(x,y), \mathsf{h}(y)) \approx \mathsf{f}(\mathsf{g}(z,z), x, x) &\Rightarrow_{[\mathsf{d}]} x \approx \mathsf{g}(z,z), \mathsf{g}(x,y) \approx x, \mathsf{h}(y) \approx x\\ &\Rightarrow_{[\mathsf{v}]} \mathsf{h}(y) \approx \mathsf{g}(z,z), \mathsf{g}(\mathsf{h}(y), y) \approx \mathsf{h}(y)\end{aligned}$$

10. (a) LPO with the precedence $\times > + > \mathsf{s}$ applies.

(b) The three overlaps

$$\begin{aligned}&\langle \mathsf{s}(\mathsf{s}(x)) \to x, 1, \mathsf{s}(y) + z \to \mathsf{s}(y+z)\rangle\\ &\langle \mathsf{s}(\mathsf{s}(x)) \to x, 1, \mathsf{s}(y) \times z \to (y \times z) + z\rangle\\ &\langle \mathsf{s}(\mathsf{s}(x)) \to x, 1, \mathsf{s}(\mathsf{s}(y)) \to y\rangle\end{aligned}$$

give the following critical pairs:

$$x + z \leftarrow\!\rtimes\!\rightarrow \mathsf{s}(\mathsf{s}(x) + z) \tag{1}$$

$$x \times z \leftarrow\!\rtimes\!\rightarrow (\mathsf{s}(x) \times z) + z \tag{2}$$

$$\mathsf{s}(x) \leftarrow\!\rtimes\!\rightarrow \mathsf{s}(x) \tag{3}$$

(c) The first critical pair is convergent because $\mathsf{s}(\mathsf{s}(x) + z)$ rewrites to $x + z$. The second critical pair is normalized to $x \times z \approx ((x \times z) + z) + z$ and subsequently oriented into

$$((x \times z) + z) + z \to x \times z$$

The third critical pair is obviously convergent. The newly added rule gives rise to two new critical pairs and it is better to wait for lecture 6 before continuing.

11. We start the completion process with the equations

$$\begin{aligned}\mathsf{f}(\mathsf{f}(x)) &\approx \mathsf{g}(x)\\ \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) &\approx \mathsf{f}(x)\end{aligned}$$

The smart thing to do is to use LPO with precedence $\mathsf{g} > \mathsf{f}$. Then the first equation is oriented into the rewrite rule

$$\mathsf{g}(x) \to \mathsf{f}(\mathsf{f}(x)) \tag{1}$$

and the second equation is simplified into

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) \approx \mathsf{f}(x)$$

and subsequently oriented into the rule

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) \to \mathsf{f}(x) \tag{2}$$

Rule (2) gives rise to three critical pairs:

$$\begin{aligned}\mathsf{f}(\mathsf{f}(x)) &\leftarrow\!\rtimes\!\rightarrow \mathsf{f}(\mathsf{f}(x))\\ \mathsf{f}(\mathsf{f}(\mathsf{f}(x))) &\leftarrow\!\rtimes\!\rightarrow \mathsf{f}(\mathsf{f}(\mathsf{f}(x)))\\ \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) &\leftarrow\!\rtimes\!\rightarrow \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x))))\end{aligned}$$

All critical pairs are convergent and hence the TRS consisting of the rules (1) and (2) is complete and has the same conversion relation as the original TRS.

12. Let us number the rewrite rules of the SRS:

$$(1)\ \mathsf{TCAT} \to \mathsf{T} \qquad (2)\ \mathsf{GAG} \to \mathsf{AG} \qquad (3)\ \mathsf{CTC} \to \mathsf{TC} \qquad (4)\ \mathsf{AGTA} \to \mathsf{A} \qquad (5)\ \mathsf{TAT} \to \mathsf{CT}$$

There are ten criticial pairs (the originating overlap is given on the right):

$$\begin{array}{rll}
\mathsf{TCAT} \leftarrow\rtimes\rightarrow \mathsf{TCAT} & \qquad & \langle(1), 111, (1)\rangle \\
\mathsf{TCACT} \leftarrow\rtimes\rightarrow \mathsf{TAT} & & \langle(1), 111, (5)\rangle \\
\mathsf{GA} \leftarrow\rtimes\rightarrow \mathsf{AGTA} & & \langle(2), 1, (4)\rangle \\
\mathsf{GAAG} \leftarrow\rtimes\rightarrow \mathsf{AGAG} & & \langle(2), 11, (2)\rangle \\
\mathsf{CT} \leftarrow\rtimes\rightarrow \mathsf{TCAT} & & \langle(3), 1, (1)\rangle \\
\mathsf{CTTC} \leftarrow\rtimes\rightarrow \mathsf{TCTC} & & \langle(3), 11, (3)\rangle \\
\mathsf{AGCT} \leftarrow\rtimes\rightarrow \mathsf{AT} & & \langle(4), 11, (5)\rangle \\
\mathsf{AGTA} \leftarrow\rtimes\rightarrow \mathsf{AGTA} & & \langle(4), 111, (4)\rangle \\
\mathsf{TAT} \leftarrow\rtimes\rightarrow \mathsf{CTCAT} & & \langle(5), 11, (1)\rangle \\
\mathsf{TACT} \leftarrow\rtimes\rightarrow \mathsf{CTAT} & & \langle(5), 11, (5)\rangle
\end{array}$$