Exercises

1. Study the slides.
2. Consider the ES $\mathcal{E}$ consisting of the single rewrite rule

   $$\mathsf{f}(\mathsf{f}(x)) \approx \mathsf{g}(x)$$

   Construct three different complete and reduced TRSs $\mathcal{R}$ such that $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$.
3. Construct a complete and reduced TRS for the ES consisting of the two equations

   $$\mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \approx x \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x))))) \approx x$$

4. Which of the critical pairs of the SRS

   $$\mathsf{TCAT} \to \mathsf{T} \qquad \mathsf{GAG} \to \mathsf{AG} \qquad \mathsf{CTC} \to \mathsf{TC} \qquad \mathsf{AGTA} \to \mathsf{A} \qquad \mathsf{TAT} \to \mathsf{CT}$$

   are reducible?
5. Show the termination of the TRS consisting of the rewrite rules

   $$\begin{aligned} \ominus(\mathsf{0}) &\to \mathsf{0} & x - \mathsf{0} &\to x & \mathsf{s}(\mathsf{p}(x)) &\to x \\ \ominus(\mathsf{s}(x)) &\to \mathsf{p}(\ominus(x)) & x - \mathsf{s}(y) &\to \mathsf{p}(x - y) & \mathsf{p}(\mathsf{s}(x)) &\to x \\ \ominus(\mathsf{p}(x)) &\to \mathsf{s}(\ominus(x)) & x - \mathsf{p}(y) &\to \mathsf{s}(x - y) & x + y &\to x - \ominus(y) \end{aligned}$$

   using KBO.
6. Show the termination of the TRS consisting of the rewrite rules

   $$\begin{aligned} \mathsf{average}(\mathsf{0},\mathsf{0}) &\to \mathsf{0} & \mathsf{average}(\mathsf{s}(x),y) &\to \mathsf{average}(x,\mathsf{s}(y)) \\ \mathsf{average}(\mathsf{0},\mathsf{s}(\mathsf{0})) &\to \mathsf{0} & \mathsf{average}(x,\mathsf{s}(\mathsf{s}(\mathsf{s}(y)))) &\to \mathsf{s}(\mathsf{average}(\mathsf{s}(x),y)) \\ \mathsf{average}(\mathsf{0},\mathsf{s}(\mathsf{s}(\mathsf{0}))) &\to \mathsf{s}(\mathsf{0}) \end{aligned}$$

   using KBO.
7. A TRS is said to be *simply terminating* if it is compatible with a reduction order $>$ that has the *subterm property*: $s > t$ whenever $s \rhd t$.
   (a) Prove that a TRS $\mathcal{R}$ is simply terminating if and only if the TRS $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is terminating. Here $\mathcal{E}\mathsf{mb}$ consists of all rewrite rules

   $$f(x_1, \ldots, x_n) \to x_i$$

   with f a function symbol of arity $n \geqslant 1$, $i \in \{1, \ldots, n\}$, and pairwise different variables $x_1, \ldots, x_n$.
   (b) Prove that TRSs whose termination can be shown using KBO are simply terminating.
   (c) Prove that polynomially terminating TRSs are simply terminating.
8. Consider the terminating TRS $\mathcal{R}$ consisting of the rewrite rules

   $$\begin{aligned} x - \mathsf{0} &\to \mathsf{0} & \mathsf{0} \div \mathsf{s}(y) &\to \mathsf{0} \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y & \mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y)) \end{aligned}$$

   (a) Prove that $\mathcal{R}$ is not polynomially terminating.
   (b) Can the termination of $\mathcal{R}$ be shown using LPO?
   (c) Can the termination of $\mathcal{R}$ be shown using KBO?

9. Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$\mathsf{f}(\mathsf{0}, y) \to \mathsf{0} \qquad\qquad \mathsf{f}(\mathsf{s}(x), y) \to \mathsf{f}(\mathsf{f}(x, y), y)$$

   (a) Compute the dependency pairs of $\mathcal{R}$.

   (b) Can the termination of $\mathcal{R}$ be shown using KBO?

   (c) Prove the termination of $\mathcal{R}$ by constructing a suitable reduction pair based on a weakly monotone interpretation in $\mathbb{N}$.

$\star$10. Repeat the previous questions for the TRS consisting of the rewrite rules

$$\begin{aligned}
\mathsf{low}(n, \mathsf{nil}) &\to \mathsf{nil} & \mathsf{if\text{-}low}(\mathsf{false}, n, m : x) &\to \mathsf{low}(n, x)\\
\mathsf{low}(n, m : x) &\to \mathsf{if\text{-}low}(m \leq n, n, m : x) & \mathsf{if\text{-}low}(\mathsf{true}, n, m : x) &\to m : \mathsf{low}(n, x)\\
\mathsf{high}(n, \mathsf{nil}) &\to \mathsf{nil} & \mathsf{if\text{-}high}(\mathsf{false}, n, m : x) &\to m : \mathsf{high}(n, x)\\
\mathsf{high}(n, m : x) &\to \mathsf{if\text{-}high}(m \leq n, n, m : x) & \mathsf{if\text{-}high}(\mathsf{true}, n, m : x) &\to \mathsf{high}(n, x)\\
\mathsf{nil} +\!\!+\, y &\to y & 0 \leq y &\to \mathsf{true}\\
(n : x) +\!\!+\, y &\to n : (x +\!\!+\, y) & \mathsf{s}(x) \leq 0 &\to \mathsf{false}\\
\mathsf{quicksort}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{s}(x) \leq \mathsf{s}(y) &\to x \leq y\\
\mathsf{quicksort}(n : x) &\to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+ (n : \mathsf{quicksort}(\mathsf{high}(n, x))) & &
\end{aligned}$$

11. Prove that the TRS consisting of the rewrite rules

$$\begin{aligned}
\mathsf{0} + y &\to y & \mathsf{0} \times y &\to \mathsf{0}\\
\mathsf{s}(x) + y &\to \mathsf{s}(x + y) & \mathsf{s}(x) \times y &\to (x \times y) + y
\end{aligned}$$

   is not polynomially terminating.

$\star$12. Let $\mathcal{R}$ be a TRS. Prove that $\mathcal{R}$ is terminating if and only if $\mathcal{R} \cup \mathsf{DP}(\mathcal{R})$ is terminating.