Some Solutions

1. Study the slides.

2. If we orient the equation $\mathsf{f}(\mathsf{f}(x)) \approx \mathsf{g}(x)$ from right to left, we obtain the complete TRS consisting of the rule

$$\mathsf{g}(x) \to \mathsf{f}(\mathsf{f}(x)) \tag{1}$$

Orienting the given equation from left to right gives the rule

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x) \tag{2}$$

The only critical pair, $\mathsf{f}(\mathsf{g}(x)) \approx \mathsf{g}(\mathsf{f}(x))$, can be oriented both ways:

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{f}(x)) \tag{3}$$

or

$$\mathsf{g}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(x)) \tag{4}$$

In both cases there is one new convergent critical pair. So we end up with the following three TRSs: $\mathcal{R}_1 = \{(1)\}$, $\mathcal{R}_2 = \{(2),(3)\}$, and $\mathcal{R}_3 = \{(2),(4)\}$. All three TRSs are reduced and terminating. The first two are compatible with LPO with respective precedences $\mathsf{g} > \mathsf{f}$ and $\mathsf{f} > \mathsf{g}$. For $\mathcal{R}_3$ we can take KBO with $w(\mathsf{f}) = w(\mathsf{g}) = w_0 = 1$ and $\mathsf{g} > \mathsf{f}$.

3. The first equation of the given ES is oriented into

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \to x \tag{1}$$

Using this rule, the second equation simplifies to $\mathsf{f}(\mathsf{f}(x)) \approx x$ and is subsequently oriented into

$$\mathsf{f}(\mathsf{f}(x)) \to x \tag{2}$$

Rule (2) simplifies rule (1) into

$$\mathsf{f}(x) \to x \tag{3}$$

Rule (3) makes rule (2) superfluous. It follows that the last rule constitutes a complete and reduced TRS for the given ES.

4. None of the ten critical pairs are reduced. The reason is simply that critical pairs of a reduced SRS are never reduced.

5. Use the weight function $w(\mathsf{0}) = w(\mathsf{s}) = w(\mathsf{p}) = w(\ominus) = w(-) = 1$ and $w(+) = 3$ together with the precedence $-, \ominus > \mathsf{s}, \mathsf{p}$.

6. Use the weight function $w(\mathsf{average}) = 0$ and $w(\mathsf{0}) = w(\mathsf{s}) = 1$ together with the empty precedence.

7. (a) Suppose $\mathcal{R}$ is simply terminating. By definition there exists a compatible reduction order $>$ with the subterm property. Because $>$ has the subterm property, $\mathcal{E}\mathsf{mb} \subseteq >$. Hence $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is terminating. Conversely, if $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is terminating then $> := \to^+_{\mathcal{R}\cup\mathcal{E}\mathsf{mb}}$ is a reduction order that is compatible with $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ and thus also with $\mathcal{R}$. From the inclusion $\to^+_{\mathcal{E}\mathsf{mb}} \subseteq >$ it easily follows that $>$ has the subterm property. Hence $\mathcal{R}$ is simply terminating.

   (b) It suffices to show that

$$f(x_1, \dots, x_n) >_{\mathsf{kbo}} x_i$$

   for every n-ary function symbol f with $n \geqslant 1$ and $i \in \{1, \dots, n\}$, and *any* precedence and admissible weight function. We distinguish two cases. If $n > 1$ or $w(f) > 0$ then $w(f(x_1, \dots, x_n)) > w(x_i)$. In the other case $n = 1$ and $w(f) = 0$, and thus $w(f(x_1, \dots, x_n)) = w(x_i)$. In this case we conclude

$$f(x_1, \dots, x_n) >_{\mathsf{kbo}} x_i$$

   by the first clause of the second case in the definition of KBO.

(c) Suppose $\mathcal{R}$ is a polynomially terminating TRS. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. So $\mathcal{R}$ is compatible with an algebra $(\mathbb{N}, \{f_\mathbb{N}\}_{f \in \mathcal{F}}, >)$ such that $f_\mathbb{N}$ is a strictly monotone polynomial function for every function symbol $f \in \mathcal{F}$. An easy induction on the value of a_i shows that $(\star)$ $f_\mathbb{N}(a_1, \ldots, a_n) \geqslant a_i$ for all n-ary function symbols f, natural numbers $a_1, \ldots, a_n$, and $i \in \{1, \ldots, n\}$. We will now construct a reduction order with the subterm property that is compatible with $\mathcal{R}$. The obvious candidate $>_\mathbb{N}$ does not work since the subterm property may not be satisfied. The trick is to extend $>_\mathbb{N}$ to a relation $\sqsupset$ as follows: $s \sqsupset t$ if

- $s >_\mathbb{N} t$, or
- $s \geqslant_\mathbb{N} t$ and $|s\sigma| > |t\sigma|$ for all substitutions σ.

It is not difficult to check that $\sqsupset$ is a reduction order with the subterm property that is compatible with $\mathcal{R}$. The proof of the subterm property of $\sqsupset$ uses $(\star)$.

8. (a) The following sequence in $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ reveals that $\mathcal{R}$ cannot be simply terminating:

$$\mathsf{s}(x) \div \mathsf{s}(\mathsf{s}(x)) \to_\mathcal{R} \mathsf{s}((x - \mathsf{s}(x)) \div \mathsf{s}(\mathsf{s}(x))) \to_{\mathcal{E}\mathsf{mb}} (x - \mathsf{s}(x)) \div \mathsf{s}(\mathsf{s}(x)) \to_{\mathcal{E}\mathsf{mb}} \mathsf{s}(x) \div \mathsf{s}(\mathsf{s}(x))$$

According to part (c) of the previous exercise, $\mathcal{R}$ is not polynomially terminating.

(b) No. Since LPO has the subterm property (cf. exercise 6 of day 2) and $\mathcal{R}$ is not simply terminating, LPO cannot be used to show the termination of $\mathcal{R}$.

(c) No. The rule $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$ is duplicating. Alternatively, one can use part (b) of the previous exercise.

9. (a) There are two dependency pairs:

$$\mathsf{f}^\sharp(\mathsf{s}(x), y) \to \mathsf{f}^\sharp(\mathsf{f}(x, y), y) \qquad \mathsf{f}^\sharp(\mathsf{s}(x), y) \to \mathsf{f}^\sharp(x, y)$$

(b) No, because the rule $\mathsf{f}(\mathsf{s}(x), y) \to \mathsf{f}(\mathsf{f}(x, y), y)$ is duplicating.

(c) Consider the reduction pair $(>_\mathbb{N}, \geqslant_\mathbb{N})$ induced by the following weakly monotone interpretation in $\mathbb{N}$: $\mathsf{0}_\mathbb{N} = 0$, $\mathsf{s}_\mathbb{N}(x) = x + 1$, and $\mathsf{f}_\mathbb{N}(x, y) = \mathsf{f}^\sharp_\mathbb{N}(x, y) = x$. Since $\mathsf{DP}(\mathcal{R}) \subseteq >_\mathbb{N}$ and $\mathcal{R} \subseteq \geqslant_\mathbb{N}$, $\mathcal{R}$ is terminating.

$\star$10. (a) No, because the rule $\mathsf{low}(n, m : x) \to \mathsf{if\text{-}low}(m \leq n, n, m : x)$ is duplicating.

(b) There are 15 dependency pairs:

$$\begin{aligned}
\mathsf{low}^\sharp(n, m : x) &\to \mathsf{if\text{-}low}^\sharp(m \leq n, n, m : x) & \mathsf{if\text{-}low}^\sharp(\mathsf{false}, n, m : x) &\to \mathsf{low}^\sharp(n, x)\\
\mathsf{low}^\sharp(n, m : x) &\to m \leq^\sharp n & \mathsf{if\text{-}low}^\sharp(\mathsf{true}, n, m : x) &\to \mathsf{low}^\sharp(n, x)\\
\mathsf{high}^\sharp(n, m : x) &\to m \leq^\sharp n & \mathsf{if\text{-}high}^\sharp(\mathsf{false}, n, m : x) &\to \mathsf{high}^\sharp(n, x)\\
\mathsf{high}^\sharp(n, m : x) &\to \mathsf{if\text{-}high}^\sharp(m \leq n, n, m : x) & \mathsf{if\text{-}high}^\sharp(\mathsf{true}, n, m : x) &\to \mathsf{high}^\sharp(n, x)\\
(n : x) +\!\!+^\sharp y &\to x +\!\!+^\sharp y & \mathsf{s}(x) \leq^\sharp \mathsf{s}(y) &\to x \leq^\sharp y
\end{aligned}$$

$$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+^\sharp (n : \mathsf{quicksort}(\mathsf{high}(n, x)))$$

$$\begin{aligned}
\mathsf{quicksort}^\sharp(n : x) &\to \mathsf{quicksort}^\sharp(\mathsf{low}(n, x)) & \mathsf{quicksort}^\sharp(n : x) &\to \mathsf{low}^\sharp(n, x)\\
\mathsf{quicksort}^\sharp(n : x) &\to \mathsf{quicksort}^\sharp(\mathsf{high}(n, x)) & \mathsf{quicksort}^\sharp(n : x) &\to \mathsf{high}^\sharp(n, x)
\end{aligned}$$

(c) The resulting constraints are satisfied by the following weakly monotone interpretation in the positive natural numbers: $\mathsf{0}_\mathbb{N} = \mathsf{false}_\mathbb{N} = \mathsf{true}_\mathbb{N} = \mathsf{nil}_\mathbb{N} = 1$, $\mathsf{s}_\mathbb{N}(x) = x + 1$, $\mathsf{low}_\mathbb{N}(x, y) = \mathsf{high}_\mathbb{N}(x, y) = y$, $\mathsf{if\text{-}low}_\mathbb{N}(b, x, y) = \mathsf{if\text{-}high}_\mathbb{N}(b, x, y) = \mathsf{if\text{-}low}^\sharp_\mathbb{N}(b, x, y) = \mathsf{if\text{-}high}^\sharp_\mathbb{N}(b, x, y) = y$, $\mathsf{low}^\sharp_\mathbb{N}(x, y) = \mathsf{high}^\sharp_\mathbb{N}(x, y) = y + 1$, $\leq_\mathbb{N}(x, y) = \leq^\sharp_\mathbb{N}(x, y) = +\!\!+^\sharp_\mathbb{N}(x, y) = x$, $:_\mathbb{N}(x, y) = x + y + 1$, $+\!\!+_\mathbb{N}(x, y) = x + y$, and $\mathsf{quicksort}_\mathbb{N}(x) = \mathsf{quicksort}^\sharp_\mathbb{N}(x) = 3^{x+1}$. (Using techniques developed in lecture 10, the termination proof becomes much simpler.)