Exercises

1. Study the slides.
2. Is the TRS consisting of the rules

   $\mathsf{f}(\mathsf{g}(x), y) \to \mathsf{f}(\mathsf{f}(\mathsf{g}(x), y), y) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{a})$

   confluent?
3. Is the TRS consisting of the rules

   $\mathsf{f}(\mathsf{g}(x), y) \to \mathsf{f}(x, \mathsf{f}(y, x)) \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{b} \to \mathsf{c}$

   confluent?
4. Prove that the SRS consisting of the rewrite rules

   $\mathsf{aba} \to \mathsf{aa} \qquad \mathsf{caa} \to \mathsf{caba}$

   is confluent.
5. Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

   $\mathsf{a} \to \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(x) \to \mathsf{g}(x, x) \qquad \mathsf{b} \to \mathsf{a} \qquad \mathsf{a} \to \mathsf{g}(\mathsf{a}, \mathsf{a}) \qquad \mathsf{g}(x, y) \to x$

   (a) Prove that $\mathcal{R}$ is confluent.

   (b) Compute the descendants of the redex a in all terms along the rewrite sequence

   $\mathsf{f}(\mathsf{a}) \to \mathsf{g}(\mathsf{a}, \mathsf{a}) \to \mathsf{a} \to \mathsf{f}(\mathsf{b})$
6. Consider the TRS consisting of the following rules:

   $\mathsf{f}(x) \to \mathsf{g}(x, x, \mathsf{a}) \qquad \mathsf{a} \to \mathsf{b}$

   Compute the descendants of the redex a along the rewrite sequence $\mathsf{f}(\mathsf{a}) \to \mathsf{g}(\mathsf{a}, \mathsf{a}, \mathsf{a}) \to \mathsf{g}(\mathsf{a}, \mathsf{b}, \mathsf{a})$.
7. Consider slide 24 of lecture 8. Prove that the confluence criteria of Huet and of Toyama are incomparable.
8. Show that the condition

   $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\to^{=} \cdot {}^{*}\!\leftarrow) \cap (\to^{*} \cdot {}^{=}\!\leftarrow)$

   is insufficient to guarantee confluence of all left-linear TRSs.

⋆9. Construct a TRS without critical pairs that is weakly normalizing but not confluent.

10. Consider the combination of two TRSs $\mathcal{R}$ and $\mathcal{S}$ over disjoint signatures $\mathcal{F}$ and $\mathcal{G}$. The *rank* of a term t in the combination is the maximum number of signature changes along any of its paths. In the example on slide 31 of lecture 8 we have $\mathrm{rank}(\mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g}(\mathsf{a}, \mathsf{b}))) = 3$ due to the path $\mathsf{f} - \mathsf{g} - \mathsf{a}$. The rank of a rewrite sequence is the rank of its initial term.

    (a) Prove that $\mathrm{rank}(s) > \mathrm{rank}(t)$ whenever $s \to t$ is a rewrite step in $\mathcal{R} \cup \mathcal{S}$.

    (b) Assume that $\mathcal{R}$ and $\mathcal{S}$ are terminating. Prove that all terms of rank 1 and 2 are terminating.

    ⋆(c) True or false? If $\mathcal{R}$ and $\mathcal{S}$ are terminating but $\mathcal{R} \cup \mathcal{S}$ is not, then there is an infinite rewrite sequence of rank 3.

11. Let $\mathcal{R}_1$ be the extension of CL with the rule

$$\mathsf{D}(x, x) \to \mathsf{E}$$

and let $\mathcal{R}_2$ be the extension of CL with the rule

$$\mathsf{D}\,x\,x \to \mathsf{E}$$

Exactly one of these extensions is confluent. Which one?

12. Give three rewrite sequences to normal form of the CL term $\mathsf{SI(Kx)Iy}$.

13. Consider the CL term $t = \mathsf{ISK(S(II)I)(SI(KII))}$.

    (a) Determine all redexes in t and their positions.

    (b) Rewrite t to normal form.

    (c) Does t admit infinite rewrite sequences?

14. For each of the following rewrite sequences $A\colon s \to^* t$ in CL, compute $p \backslash A$ for every position p in s.

    (a) $\mathsf{SKII} \to \mathsf{KI(II)} \to \mathsf{I}$

    (b) $\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$

    (c) $\mathsf{SSSSSS} \to \mathsf{SS(SS)SS} \to \mathsf{SS(SSS)S} \to \mathsf{SS(SSSS)} \to \mathsf{SS(SS(SS))}$

15. Compute all rewrite sequences starting from the term $\mathsf{s(0)} \times (\mathsf{0} + \mathsf{s(0)})$ in the TRS consisting of the rewrite rules

$$\mathsf{0} + y \to y \qquad\qquad \mathsf{0} \times y \to \mathsf{0}$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

Which of these are (complete) developments?

16. Compute all CL rewrite sequences between the combinators $\mathsf{K(I(II))(II)}$ and KII and determine which are permutation equivalent.

17. Suppose we extend CL with constants P and $\underline{n}$ for every natural number n and rewrite rules

$$\mathsf{P}\,\underline{m}\,\underline{n} \to \underline{m+n}$$

for all natural numbers m and n. Normalize the term $\mathsf{S(S(KP)(SPI))(SPI)(SP(SPI)}\underline{2}\mathsf{)}$ using the leftmost innermost and full-substitution strategies.

18. Determine in which of the rewrite sequences computed in exercise 15 only needed redexes are contracted.

19. Prove that every reducible term in an orthogonal TRS has an outermost redex that is needed.

20. Let $s \to t$ be a rewrite step in an orthogonal TRS. Exactly one of the following statements is true.

    (a) Descendants in t of needed redexes in s are needed.

    (b) Descendants in t of non-needed redexes in s are non-needed.

    Which is true? Give a proof and a counterexample.

21. Consider the example on slide 23 of lecture 9.

    (a) Compute $\text{normalize}_A(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty)))$.

    (b) Give a full strategy annotation B such that $\text{normalize}_B(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty))) \neq \mathsf{T}$.