Solutions

1 For $\mathcal{A}_1$ we can take the ARS

$$\mathsf{a} \longleftarrow \mathsf{b} \rightleftarrows \mathsf{c} \longrightarrow \mathsf{d}$$

For $\mathcal{A}_2$ we can take the ARS

$$\mathsf{a} \longleftarrow \mathsf{b} \rightleftarrows \mathsf{c} \longrightarrow \mathsf{d} \circlearrowleft$$

ARS $\mathcal{A}_3$ does not exist because every ARS with the properties WN and UN must satisfy CR as well.

2 (a) For instance,

$$\begin{aligned}\mathsf{g}(\mathsf{c}(\mathsf{f}(\mathsf{a}))) &\to \mathsf{g}(\mathsf{c}(\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{f}(\mathsf{c}(\mathsf{a}))) \to \mathsf{f}(\mathsf{c}(\mathsf{c}(\mathsf{f}(\mathsf{a})))) \to \mathsf{f}(\mathsf{c}(\mathsf{c}(\mathsf{a})))\\ &\to \mathsf{c}(\mathsf{c}(\mathsf{f}(\mathsf{c}(\mathsf{a})))) \to \mathsf{c}(\mathsf{c}(\mathsf{c}(\mathsf{c}(\mathsf{f}(\mathsf{a}))))) \to \mathsf{c}(\mathsf{c}(\mathsf{c}(\mathsf{c}(\mathsf{a}))))\end{aligned}$$

(b) Yes, because $\mathcal{R}$ is orthogonal.

(c) The polynomial interpretation $\mathsf{a}_{\mathbb{N}} = 1$, $\mathsf{c}_{\mathbb{N}}(x) = x + 3$, $\mathsf{f}_{\mathbb{N}}(x) = 3x + 1$, and $\mathsf{g}_{\mathbb{N}}(x) = 14x$ orients the rewrite rules of $\mathcal{R}$ from left to right:

$$4 > 1 \qquad\qquad 14 > 13$$
$$3x + 10 > 3x + 7 \qquad\qquad 56 > 43$$
$$14x + 84 > 14x + 9$$

for all $x \in \mathbb{N}$.

(d) Yes. If we take the precedence $\mathsf{g} > \mathsf{f} > \mathsf{c}$ then $l >_{\mathsf{lpo}} r$ for all rewrite rules $l \to r \in \mathcal{R}$.

(e) No. The rewrite rule $\mathsf{f}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{c}(\mathsf{f}(x)))$ requires $w(\mathsf{c}) = 0$. Hence the rule $\mathsf{g}(\mathsf{c}(\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}))$ requires $w(\mathsf{f}) = 0$. However, we cannot have different unary function symbols of weight 0 because KBO can only be used to prove termination if every unary function symbol of weight 0 is greater than any other function symbol in the precedence (*admissibility*).

3 (a) For instance, $\mathsf{f}(\mathsf{f}(x)) >_{\mathsf{lpo}} \mathsf{g}(\mathsf{g}(\mathsf{h}(x)))$ for the precedence $\mathsf{f} \succ \mathsf{g}, \mathsf{h}$.

(b) There is only one critical pair, $\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{h}(x)))) \leftarrow\!\rtimes\!\rightarrow \mathsf{g}(\mathsf{g}(\mathsf{h}(\mathsf{f}(x))))$, stemming from the overlap $\langle \mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{h}(x))), 1, \mathsf{f}(\mathsf{f}(y)) \to \mathsf{g}(\mathsf{g}(\mathsf{h}(y)))\rangle$. Since the critical pair consists of different normal forms, it is not convergent.

(c) Adding the rewrite rule

$$\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{h}(x)))) \to \mathsf{g}(\mathsf{g}(\mathsf{h}(\mathsf{f}(x)))) \tag{2}$$

to $\mathcal{R}$ makes the critical pair of part (b) convergent while preserving termination. There is one new critical pair: $\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{h}(\mathsf{f}(x))))) \leftarrow\!\rtimes\!\rightarrow \mathsf{g}(\mathsf{g}(\mathsf{h}(\mathsf{g}(\mathsf{g}(\mathsf{h}(x))))))$. The left-hand side rewrites in two step to the right-hand side. Hence the critical pair is convergent. A different complete reduced TRS is obtained by reversing rewrite rule (2):

$$\mathsf{g}(\mathsf{g}(\mathsf{h}(\mathsf{f}(x)))) \to \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{h}(x)))) \tag{3}$$

Again the single new critical pair is convergent. Termination of this TRS can be shown by the following polynomial interpretation: $\mathsf{f}_{\mathbb{N}}(x) = 3x + 1$ and $\mathsf{g}_{\mathbb{N}}(x) = \mathsf{h}_{\mathbb{N}}(x) = 2x$. These are not the only complete reduced TRSs with the same conversion as $\mathcal{R}$. For instance, completion with respect to LPO with precedence $\mathsf{g} > \mathsf{f}$ results in the complete reduced TRS consisting of the single rule

$$\mathsf{g}(\mathsf{g}(\mathsf{h}(x))) \to \mathsf{f}(\mathsf{f}(x)) \tag{4}$$

4

(a) We have $\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII)} \to \mathsf{SII(SII)}$ and thus t admits an infinite rewrite sequence as it contains $\mathsf{SII(SII)}$ as subterm.

(b) The leftmost outermost strategy produces the following rewrite sequence:

$$t \to \mathsf{K(KI(SII(SII)))(I(KI(SII(SII))))} \to \mathsf{KI(SII(SII))} \to \mathsf{I}$$

(c) The full substitution strategy produces the following rewrite sequence:

$$t \twoheadrightarrow \mathsf{KI(II)} \twoheadrightarrow \mathsf{I}$$