ISR 2010

Introduction to Term Rewriting

lecture 10

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Matrix Interpretations
- Dependency Pairs
- Semantic Labeling
- Derivational Complexity
- Further Reading

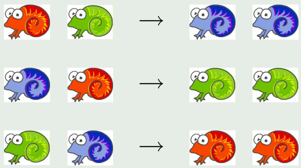
Example

SRS

is non-terminating

Example

SRS



is terminating

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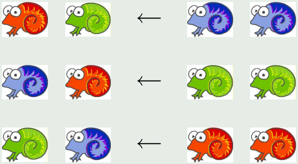

is terminating ? (RTA open problem # 104)

Example

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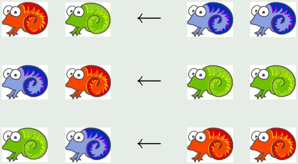

is terminating ! (RTA open problem # 104)

Definition

algebra $\mathcal{M}$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$
- interpretations (for every n-ary f)

$$f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = M_1\vec{x}_1 + \cdots + M_n\vec{x}_n + \vec{f}$$

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- $(x_1, \ldots, x_d)^{\mathsf{T}} > (y_1, \ldots, y_d)^{\mathsf{T}} \iff x_1 > y_1 \ \wedge \ \bigwedge_{i=2}^{d} x_i \geqslant y_i$

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Lemma

$(\mathcal{M}, >)$ *is well-founded monotone algebra*

Example (1)

rewrite rules

$$\begin{aligned} \mathsf{a}(\mathsf{a}(x)) &\rightarrow \mathsf{b}(\mathsf{c}(x)) \\ \mathsf{b}(\mathsf{b}(x)) &\rightarrow \mathsf{a}(\mathsf{c}(x)) \\ \mathsf{c}(\mathsf{c}(x)) &\rightarrow \mathsf{a}(\mathsf{b}(x)) \end{aligned}$$

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rewrite rules

$$\begin{aligned} a(a(x)) &\rightarrow b(c(x)) \\ b(b(x)) &\rightarrow a(c(x)) \\ c(c(x)) &\rightarrow a(b(x)) \end{aligned}$$

matrix interpretation

$$a_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$b_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$c_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

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$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

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$$\mathsf{c}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\forall \vec{x} \quad \begin{aligned} \mathsf{a}_{\mathcal{M}}(\mathsf{a}_{\mathcal{M}}(\vec{x})) &> \mathsf{b}_{\mathcal{M}}(\mathsf{c}_{\mathcal{M}}(\vec{x})) \\ \mathsf{b}_{\mathcal{M}}(\mathsf{b}_{\mathcal{M}}(\vec{x})) &> \mathsf{a}_{\mathcal{M}}(\mathsf{c}_{\mathcal{M}}(\vec{x})) \\ \mathsf{c}_{\mathcal{M}}(\mathsf{c}_{\mathcal{M}}(\vec{x})) &> \mathsf{a}_{\mathcal{M}}(\mathsf{b}_{\mathcal{M}}(\vec{x})) \end{aligned}$$

Example (2)

rewrite rules

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{c}) \qquad \mathsf{h}(\mathsf{c}) \to \mathsf{h}(\mathsf{a})$$

Example (2)

rewrite rules

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matrix interpretation

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{a}_{\mathcal{M}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

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Example (3)

rewrite rules

$$\mathsf{f}(\mathsf{f}(x)) \rightarrow \mathsf{g}(x)$$
$$\mathsf{g}(\mathsf{h}(x)) \rightarrow \mathsf{f}(\mathsf{h}(\mathsf{f}(x)))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) \rightarrow \mathsf{i}(x, x, x)$$

$$\mathsf{i}(\mathsf{a}_1, x, \mathsf{b}_1) \rightarrow \mathsf{i}(x, \mathsf{a}_2, \mathsf{b}_1)$$
$$\mathsf{i}(x, \mathsf{a}_3, \mathsf{b}_1) \rightarrow \mathsf{i}(\mathsf{a}_4, x, \mathsf{b}_1)$$
$$\mathsf{i}(x, x, \mathsf{b}_1) \rightarrow \mathsf{i}(\mathsf{g}(x), \mathsf{b}_2, \mathsf{b}_2)$$
$$\mathsf{i}(\mathsf{g}(x), \mathsf{b}_3, \mathsf{b}_3) \rightarrow \mathsf{i}(x, x, \mathsf{b}_4)$$

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$$\mathsf{g}(\mathsf{h}(x)) \to \mathsf{f}(\mathsf{h}(\mathsf{f}(x))) \qquad \mathsf{i}(x, \mathsf{a}_3, \mathsf{b}_1) \to \mathsf{i}(\mathsf{a}_4, x, \mathsf{b}_1)$$
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matrix interpretation

$$\mathsf{f}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathsf{a}_{1\mathcal{A}} = \mathsf{a}_{3\mathcal{A}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathsf{b}_{1\mathcal{A}} = \begin{pmatrix} 13 \\ 0 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{a}_{2\mathcal{A}} = \mathsf{a}_{4\mathcal{A}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \mathsf{b}_{2\mathcal{A}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\mathsf{h}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{3\mathcal{A}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{4\mathcal{A}} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$\mathsf{i}_{\mathcal{A}}(\vec{x}, \vec{y}, \vec{z}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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polynomial interpretation over $\mathbb{R}$

$$\mathsf{f}_{\mathcal{A}}(x) = \sqrt{2}x + 1$$
$$\mathsf{g}_{\mathcal{A}}(x) = 2x$$
$$\mathsf{h}_{\mathcal{A}}(x) = x + 5$$
$$\mathsf{i}_{\mathcal{A}}(x, y, z) = x + y + z$$
$$\mathsf{a}_{1\mathcal{A}} = \mathsf{a}_{3\mathcal{A}} = \mathsf{b}_{1\mathcal{A}} = \mathsf{b}_{3\mathcal{A}} = 1$$
$$\mathsf{a}_{2\mathcal{A}} = \mathsf{a}_{4\mathcal{A}} = \mathsf{b}_{2\mathcal{A}} = \mathsf{b}_{4\mathcal{A}} = 0$$

Outline

Definition

DP problem is pair of finite TRSs $(\mathcal{P}, \mathcal{R})$ such that root symbols of rules in $\mathcal{P}$

- do not occur in $\mathcal{R}$
- do not occur in proper subterms of left- and right-hand sides of rules in $\mathcal{P}$

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DP problem $(\mathcal{P}, \mathcal{R})$ is finite if $\neg\,\exists$ infinite sequence

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Theorem

finite TRS $\mathcal{R}$ *is terminating* $\iff$ *DP problem* $(DP(\mathcal{R}), \mathcal{R})$ *is finite*

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DP Processors

dependency graph, increasing interpretations, instantiation, loop detection, match-bounds, narrowing, reduction pairs, rewriting, root-labeling, rule removal, size-change principle, semantic labeling, string reversal, subterm criterion, uncurrying, usable rules, . . .

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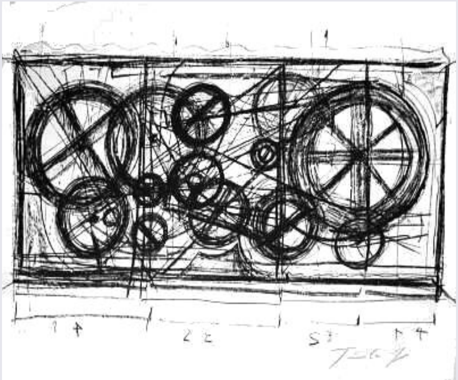
Termination Tools



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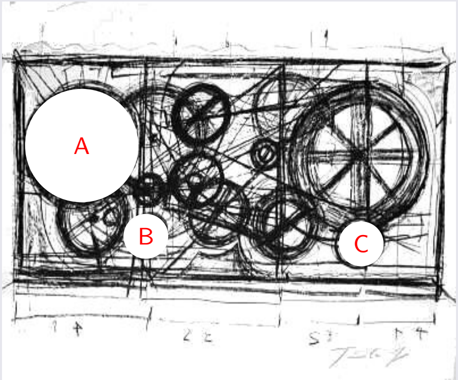

DP processors

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dependency graph $\mathrm{DG}(\mathcal{P}, \mathcal{R})$ of DP problem $(\mathcal{P}, \mathcal{R})$ consists of

- nodes: rules in $\mathcal{P}$
- arrows: $s \to t \longrightarrow u \to v$ if $\exists$ substitutions σ, τ such that $t\sigma \xrightarrow[\mathcal{R}]{*} u\tau$

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Trivial Approximation

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Trivial Approximation

$$s \to t \xrightarrow{\mathsf{t}} u \to v \quad \iff \quad \mathrm{root}(t) = \mathrm{root}(u)$$

Better Approximations

are based on

- unification ▸ details
- tree automata techniques

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Question

how to prove finiteness of DP problem $(\mathcal{P}, \mathcal{R})$?

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how to prove finiteness of DP problem $(\mathcal{P}, \mathcal{R})$?

Answers

- use pair of orders
- use minimality together with subterm relation
- ...

Definition

DP problem $(\mathcal{P}, \mathcal{R})$ is finite if $\neg\, \exists$ infinite sequence

$$t_1 \;\gtrsim\; t_2 \;>\; t_3 \;\gtrsim\; t_4 \;>\; \cdots$$

such that all terms t_1, t_2, ... are terminating with respect to $\mathcal{R}$

Question

how to prove finiteness of DP problem $(\mathcal{P}, \mathcal{R})$?

Answers

- use pair of orders $\gtrsim$ and $>$ such that $\mathcal{R} \subseteq \gtrsim$ and $\mathcal{P} \subseteq >$
- use minimality together with subterm relation
- ...

Definition

reduction pair $(>, \gtrsim)$ consists of well-founded order $>$ and preorder $\gtrsim$ such that

1. $>$ is closed under substitutions
2. $\gtrsim$ is closed under contexts and substitutions
3. $> \cdot \gtrsim \subseteq >$ or $\gtrsim \cdot > \subseteq >$

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Definition (Reduction Pair Processor)

reduction pair $(>, \gtrsim)$

$$(\mathcal{P}, \mathcal{R}) \mapsto \begin{cases} \{(\mathcal{P} \setminus >, \mathcal{R})\} & \text{if } \mathcal{P} \subseteq > \cup \gtrsim \text{ and } \mathcal{R} \subseteq \gtrsim \\ \{(\mathcal{P}, \mathcal{R})\} & \text{otherwise} \end{cases}$$

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reduction pair processor is sound and complete

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Theorem

rule removal processor is sound and complete

Example

rewrite rules

$$\begin{aligned} 0 + y &\to y \\ s(x) + y &\to s(x + y) \end{aligned}$$

dependency pairs

$$s(x) +^{\sharp} y \to x +^{\sharp} y$$

Example

rewrite rules

$$\begin{aligned} 0 + y &\rightarrow y \\ \mathsf{s}(x) + y &\rightarrow \mathsf{s}(x + y) \end{aligned}$$

dependency pairs

$$\mathsf{s}(x) +^\sharp y \rightarrow x +^\sharp y$$

polynomial interpretations

$$0_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad +_{\mathbb{N}}(x, y) = +^\sharp_{\mathbb{N}}(x, y) = 2x + y$$

Example

rewrite rules

$$0 + y \geqslant_{\mathbb{N}} y$$
$$\mathsf{s}(x) + y >_{\mathbb{N}} \mathsf{s}(x + y)$$

dependency pairs

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polynomial interpretations

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Example

rewrite rules

$$0 + y \to y$$
$$s(x) + y \to s(x + y)$$

dependency pairs

polynomial interpretations

$$0_{\mathbb{N}} = 0 \qquad s_{\mathbb{N}}(x) = x + 1 \qquad +_{\mathbb{N}}(x, y) = +^{\sharp}_{\mathbb{N}}(x, y) = 2x + y$$

reduction pair processor

Example

rewrite rules

$$0 + y \geqslant_{\mathbb{N}} y$$
$$\mathsf{s}(x) + y >_{\mathbb{N}} \mathsf{s}(x + y)$$

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rule removal processor

Outline

Remark

traditional simplification orders like LPO and KBO give rise to monotonic reduction pairs

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Definition

- argument filter is mapping π such that for every n-ary function symbol f

$$\pi(f) \in \{1, \ldots, n\} \quad \text{or} \quad \pi(f) = [i_1, \ldots, i_m] \text{ with } 1 \leqslant i_1 < \cdots < i_m \leqslant n$$

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- extension to terms:

$$\pi(t) = \begin{cases} t & \text{if } t \text{ is variable} \\ \pi(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } \pi(f) = i \\ f(\pi(t_{i_1}), \ldots, \pi(t_{i_m})) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } \pi(f) = [i_1, \ldots, i_m] \end{cases}$$

Notation

- $s >_\pi t \iff \pi(s) > \pi(t)$
- $s \gtrsim_\pi t \iff \pi(s) \gtrsim \pi(t)$

Lemma

$(>_\pi, \gtrsim_\pi)$ is reduction pair for every reduction pair $(>, \gtrsim)$ and argument filter π

Notation

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Lemma

$(>_\pi, \gtrsim_\pi)$ is reduction pair for every reduction pair $(>, \gtrsim)$ and argument filter π

Example

rewrite rules

$$\begin{aligned} 0 - y &\to 0 & 0 \div \mathsf{s}(y) &\to 0 \\ x - 0 &\to x & \mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y)) \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

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Example

DP problem

$$\begin{aligned} 0 - y &\to 0 & 0 \div \mathsf{s}(y) &\to 0 \\ x - 0 &\to x & \mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y)) \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y & \mathsf{s}(x) \div^\sharp \mathsf{s}(y) &\to (x - y) \div^\sharp \mathsf{s}(y) \end{aligned}$$

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Example

DP problem

$$
\begin{array}{rcl@{\qquad}rcl}
0 - y & \rightarrow & 0 & 0 \div \mathsf{s}(y) & \rightarrow & 0 \\
x - 0 & \rightarrow & x & \mathsf{s}(x) \div \mathsf{s}(y) & \rightarrow & \mathsf{s}((x - y) \div \mathsf{s}(y)) \\
\mathsf{s}(x) - \mathsf{s}(y) & \rightarrow & x - y & \mathsf{s}(x) \div^\sharp \mathsf{s}(y) & \rightarrow & (x - y) \div^\sharp \mathsf{s}(y)
\end{array}
$$

argument filter $\pi(-) = 1$

Notation

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$(>_\pi, \gtrsim_\pi)$ is reduction pair for every reduction pair $(>, \gtrsim)$ and argument filter π

Example

DP problem

$$\begin{array}{rclcrcl} 0 & \to & 0 & \quad & 0 \div \mathsf{s}(y) & \to & 0 \\ x & \to & x & & \mathsf{s}(x) \div \mathsf{s}(y) & \to & \mathsf{s}(x \div \mathsf{s}(y)) \\ \mathsf{s}(x) & \to & x & & \mathsf{s}(x) \div^\sharp \mathsf{s}(y) & \to & x \div^\sharp \mathsf{s}(y) \end{array}$$

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Lemma

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Example

DP problem

$$0 \;>^{=}_{\mathsf{lpo}}\; 0 \qquad 0 \div \mathsf{s}(y) \;>^{=}_{\mathsf{lpo}}\; 0$$

$$x \;>^{=}_{\mathsf{lpo}}\; x \qquad \mathsf{s}(x) \div \mathsf{s}(y) \;>^{=}_{\mathsf{lpo}}\; \mathsf{s}(\;x \qquad \div\, \mathsf{s}(y))$$

$$\mathsf{s}(x) \;>^{=}_{\mathsf{lpo}}\; x \qquad \mathsf{s}(x) \div^\sharp \mathsf{s}(y) \;>_{\mathsf{lpo}}\; x \qquad \div^\sharp \mathsf{s}(y)$$

argument filter $\pi(-) = 1$ LPO with precedence $\div > \mathsf{s}$

Outline

Definition

DP problem $(\mathcal{P}, \mathcal{R})$ is finite if $\neg\ \exists$ infinite sequence

$$t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathcal{P}]{\epsilon} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathcal{P}]{\epsilon} \cdots$$

such that all terms t_1, t_2, ... are terminating with respect to $\mathcal{R}$

Question

how to prove finiteness of DP problem $(\mathcal{P}, \mathcal{R})$?

Answers

- use pair of orders
- use minimality together with subterm relation
- ...

Idea

project each dependency pair symbol in $\mathcal{P}$ to fixed argument position

$$\pi(t_1) \xrightarrow[\mathcal{R}]{*} \pi(t_2) \quad ? \quad \pi(t_3) \xrightarrow[\mathcal{R}]{*} \pi(t_4) \quad ? \quad \cdots$$

Idea

project each dependency pair symbol in $\mathcal{P}$ to fixed argument position

$$\pi(t_1) \xrightarrow[\mathcal{R}]{*} \pi(t_2) \quad ? \quad \pi(t_3) \xrightarrow[\mathcal{R}]{*} \pi(t_4) \quad ? \quad \cdots$$

Observation

$\pi(t_1)$ is terminating with respect to $\xrightarrow[\mathcal{R}]{}$

Idea

project each dependency pair symbol in $\mathcal{P}$ to fixed argument position

$$\pi(t_1) \xrightarrow[\mathcal{R}]{*} \pi(t_2) \quad ? \quad \pi(t_3) \xrightarrow[\mathcal{R}]{*} \pi(t_4) \quad ? \quad \cdots$$

Observation

$\pi(t_1)$ is terminating with respect to $\underset{\mathcal{R}}{\longrightarrow} \cup \rhd$

Idea

project each dependency pair symbol in $\mathcal{P}$ to fixed argument position

$$\pi(t_1) \xrightarrow[\mathcal{R}]{*} \pi(t_2) \quad ? \quad \pi(t_3) \xrightarrow[\mathcal{R}]{*} \pi(t_4) \quad ? \quad \cdots$$

Observation

$\pi(t_1)$ is terminating with respect to $\xrightarrow[\mathcal{R}]{} \cup \rhd$

Definition

- simple projection for DP problem $(\mathcal{P}, \mathcal{R})$ is mapping π that assigns to every dependency pair symbol $f^\sharp$ in $\mathcal{P}$ one of its argument positions

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project each dependency pair symbol in $\mathcal{P}$ to fixed argument position

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Definition

- simple projection for DP problem $(\mathcal{P}, \mathcal{R})$ is mapping π that assigns to every dependency pair symbol $f^\sharp$ in $\mathcal{P}$ one of its argument positions
- extension to terms in $\mathcal{T}^\sharp$: $\pi(f^\sharp(t_1, \ldots, t_n)) = t_{\pi(f^\sharp)}$

Definition (Subterm Criterion Processor)

simple projection π

$$(\mathcal{P}, \mathcal{R}) \mapsto \begin{cases} \{(\{\ell \to r \in \mathcal{P} \mid \pi(\ell) = \pi(r)\}, \mathcal{R})\} & \text{if } \pi(\mathcal{P}) \subseteq \trianglerighteq \\ \{(\mathcal{P}, \mathcal{R})\} & \text{otherwise} \end{cases}$$

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Theorem

subterm criterion processor is sound and complete

Example

rewrite rules

$$\begin{aligned} \mathsf{ack}(0, y) &\to \mathsf{s}(y) \\ \mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

Example

rewrite rules

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dependency pairs

$$\begin{aligned} \mathsf{ack}^\sharp(\mathsf{s}(x), 0) &\to \mathsf{ack}^\sharp(x, \mathsf{s}(0)) \\ \mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}^\sharp(x, \mathsf{ack}(\mathsf{s}(x), y)) \\ \mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}^\sharp(\mathsf{s}(x), y) \end{aligned}$$

Example

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simple projection

$$\pi(\mathsf{ack}^\sharp) = 1$$

Example

rewrite rules

$$\begin{aligned} \mathsf{ack}(0, y) &\to \mathsf{s}(y) \\ \mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

dependency pairs

$$\begin{array}{ccc} \mathsf{s}(x) & \rhd & x \\ \mathsf{s}(x) & \rhd & x \\ \mathsf{s}(x) & \unrhd & \mathsf{s}(x) \end{array}$$

simple projection

$$\pi(\mathsf{ack}^\sharp) = 1$$

Example

rewrite rules

$$\begin{aligned} \mathsf{ack}(0, y) &\to \mathsf{s}(y) \\ \mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

dependency pairs

$$\mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}^\sharp(\mathsf{s}(x), y)$$

Example

rewrite rules

$$\begin{aligned} \mathsf{ack}(0, y) &\rightarrow \mathsf{s}(y) \\ \mathsf{ack}(\mathsf{s}(x), 0) &\rightarrow \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\rightarrow \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

dependency pairs

$$\mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \rightarrow \mathsf{ack}^\sharp(\mathsf{s}(x), y)$$

simple projection

$$\pi(\mathsf{ack}^\sharp) = 2$$

Example

rewrite rules

$$\mathsf{ack}(0, y) \to \mathsf{s}(y)$$
$$\mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$
$$\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

dependency pairs

$$\mathsf{s}(y) \quad \rhd \qquad\qquad y$$

simple projection

$$\pi(\mathsf{ack}^\sharp) = 2$$

Example

rewrite rules

$$\begin{aligned}
\mathsf{low}(n,\mathsf{nil}) &\to \mathsf{nil} & \mathsf{if\text{-}low}(\mathsf{false},n,m:x) &\to \mathsf{low}(n,x)\\
\mathsf{low}(n,m:x) &\to \mathsf{if\text{-}low}(m \leq n,n,m:x) & \mathsf{if\text{-}low}(\mathsf{true},n,m:x) &\to m:\mathsf{low}(n,x)\\
\mathsf{high}(n,\mathsf{nil}) &\to \mathsf{nil} & \mathsf{if\text{-}high}(\mathsf{false},n,m:x) &\to m:\mathsf{high}(n,x)\\
\mathsf{high}(n,m:x) &\to \mathsf{if\text{-}high}(m \leq n,n,m:x) & \mathsf{if\text{-}high}(\mathsf{true},n,m:x) &\to \mathsf{high}(n,x)\\
\mathsf{nil} \mathbin{+\!\!+} y &\to y & 0 \leq y &\to \mathsf{true}\\
(n:x) \mathbin{+\!\!+} y &\to n:(x \mathbin{+\!\!+} y) & \mathsf{s}(x) \leq 0 &\to \mathsf{false}\\
\mathsf{quicksort}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{s}(x) \leq \mathsf{s}(y) &\to x \leq y\\
\mathsf{quicksort}(n:x) &\to \mathsf{quicksort}(\mathsf{low}(n,x)) \mathbin{+\!\!+} (n:\mathsf{quicksort}(\mathsf{high}(n,x)))
\end{aligned}$$

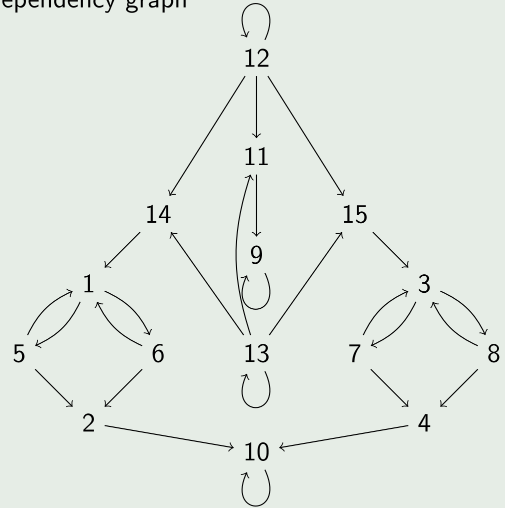
Example (cont'd)

dependency pairs

1	$\mathsf{low}^\sharp(n, m : x) \to \mathsf{if\text{-}low}^\sharp(m \leq n, n, m : x)$	5	$\mathsf{if\text{-}low}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
2	$\mathsf{low}^\sharp(n, m : x) \to m \leq^\sharp n$	6	$\mathsf{if\text{-}low}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
3	$\mathsf{high}^\sharp(n, m : x) \to \mathsf{if\text{-}high}^\sharp(m \leq n, n, m : x)$	7	$\mathsf{if\text{-}high}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{high}^\sharp(n, x)$
4	$\mathsf{high}^\sharp(n, m : x) \to m \leq^\sharp n$	8	$\mathsf{if\text{-}high}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{high}^\sharp(n, x)$
9	$(n : x) +\!\!+^\sharp y \to x +\!\!+^\sharp y$	10	$\mathsf{s}(x) \leq^\sharp \mathsf{s}(y) \to x \leq^\sharp y$
11	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+^\sharp (n : \mathsf{quicksort}(\mathsf{high}(n, x)))$		
12	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}^\sharp(\mathsf{low}(n, x))$	14	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{low}^\sharp(n, x)$
13	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}^\sharp(\mathsf{high}(n, x))$	15	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{high}^\sharp(n, x)$

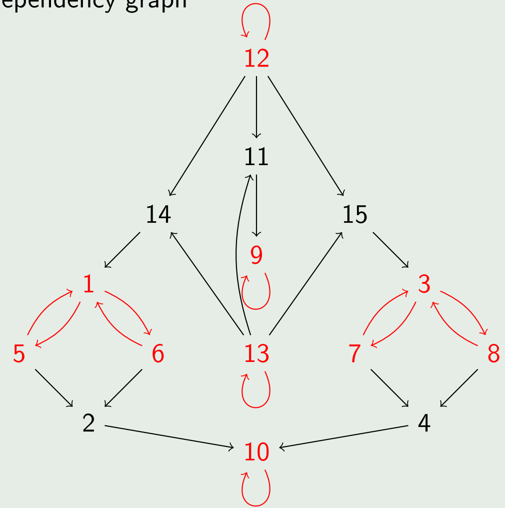
Example (cont'd)

dependency graph



Example (cont'd)

dependency graph

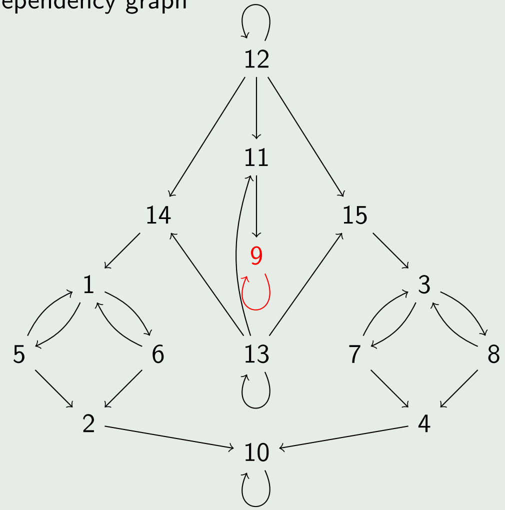


six SCCs

- $\{9\}$
- $\{10\}$
- $\{12\}$
- $\{13\}$
- $\{1, 5, 6\}$
- $\{3, 7, 8\}$

Example (cont'd)

dependency graph



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- $\{1, 5, 6\}$
- $\{3, 7, 8\}$

Example (cont'd)

dependency pairs

1	$\mathsf{low}^\sharp(n, m : x) \to \mathsf{if\text{-}low}^\sharp(m \le n, n, m : x)$	5	$\mathsf{if\text{-}low}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
2	$\mathsf{low}^\sharp(n, m : x) \to m \le^\sharp n$	6	$\mathsf{if\text{-}low}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
3	$\mathsf{high}^\sharp(n, m : x) \to \mathsf{if\text{-}high}^\sharp(m \le n, n, m : x)$	7	$\mathsf{if\text{-}high}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{high}^\sharp(n, x)$
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9	$(n : x) +\!\!+^\sharp y \to x +\!\!+^\sharp y$	10	$\mathsf{s}(x) \le^\sharp \mathsf{s}(y) \to x \le^\sharp y$
11	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+^\sharp (n : \mathsf{quicksort}(\mathsf{high}(n, x)))$		
12	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}^\sharp(\mathsf{low}(n, x))$	14	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{low}^\sharp(n, x)$
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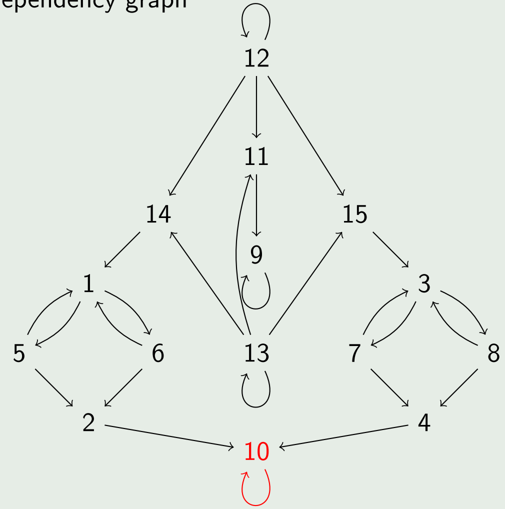
Example (cont'd)

dependency graph



six SCCs

- {9}
- {10}
- {12}
- {13}
- {1, 5, 6}
- {3, 7, 8}

subterm criterion

Example (cont'd)

dependency pairs

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2	$\mathsf{low}^\sharp(n, m : x) \to m \leq^\sharp n$	6	$\mathsf{if\text{-}low}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
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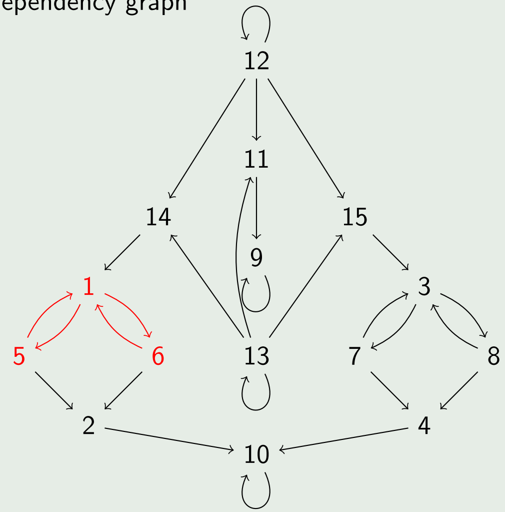
Example (cont'd)

dependency graph



six SCCs

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- $\{10\}$ subterm criterion
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Example (cont'd)

dependency pairs

1	$\mathsf{low}^\sharp(n, m : x) \to \mathsf{if\text{-}low}^\sharp(m \leq n, n, m : x)$	5	$\mathsf{if\text{-}low}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
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9	$(n : x) +\!\!+^\sharp y \to x +\!\!+^\sharp y$	10	$\mathsf{s}(x) \leq^\sharp \mathsf{s}(y) \to x \leq^\sharp y$
11	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+^\sharp (n : \mathsf{quicksort}(\mathsf{high}(n, x)))$		
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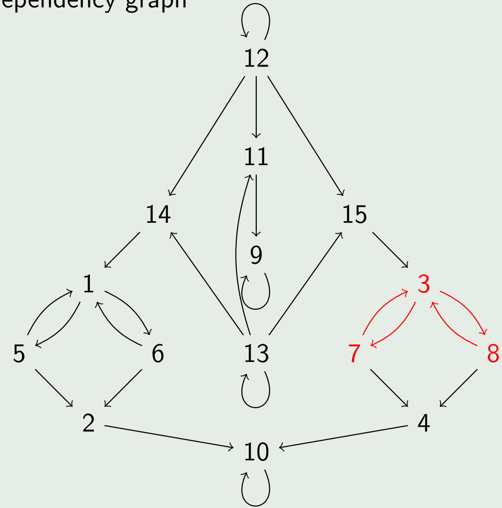
Example (cont'd)

dependency graph



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- $\{13\}$
- $\{1, 5, 6\}$ subterm criterion
- $\{3, 7, 8\}$ subterm criterion

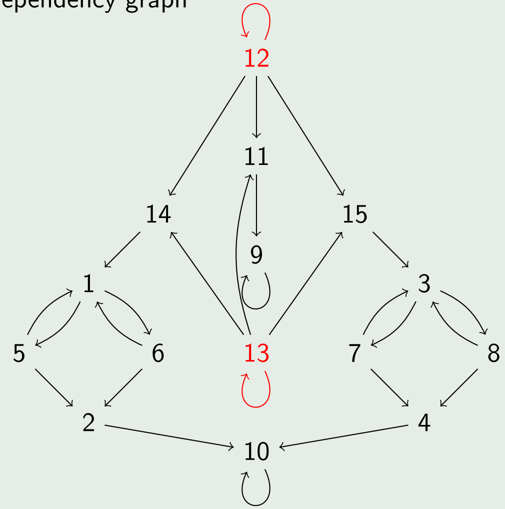
Example (cont'd)

dependency pairs

1	$\mathsf{low}^\sharp(n, m : x) \to \mathsf{if\text{-}low}^\sharp(m \leq n, n, m : x)$	5	$\mathsf{if\text{-}low}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
2	$\mathsf{low}^\sharp(n, m : x) \to m \leq^\sharp n$	6	$\mathsf{if\text{-}low}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
3	$\mathsf{high}^\sharp(n, m : x) \to \mathsf{if\text{-}high}^\sharp(m \leq n, n, m : x)$	7	$\mathsf{if\text{-}high}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{high}^\sharp(n, x)$
4	$\mathsf{high}^\sharp(n, m : x) \to m \leq^\sharp n$	8	$\mathsf{if\text{-}high}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{high}^\sharp(n, x)$
9	$(n : x) +\!\!+^\sharp y \to x +\!\!+^\sharp y$	10	$\mathsf{s}(x) \leq^\sharp \mathsf{s}(y) \to x \leq^\sharp y$
11	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+^\sharp (n : \mathsf{quicksort}(\mathsf{high}(n, x)))$		
12	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}^\sharp(\mathsf{low}(n, x))$	14	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{low}^\sharp(n, x)$
13	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}^\sharp(\mathsf{high}(n, x))$	15	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{high}^\sharp(n, x)$

Example (cont'd)

dependency graph



six SCCs

- $\{9\}$ subterm criterion
- $\{10\}$ subterm criterion
- $\{12\}$
- $\{13\}$
- $\{1, 5, 6\}$ subterm criterion
- $\{3, 7, 8\}$ subterm criterion

Example

DP problem

$$\begin{aligned}
\mathsf{low}(n, \mathsf{nil}) &\to \mathsf{nil} & \mathsf{if\text{-}low}(\mathsf{false}, n, m : x) &\to \mathsf{low}(n, x)\\
\mathsf{low}(n, m : x) &\to \mathsf{if\text{-}low}(m \le n, n, m : x) & \mathsf{if\text{-}low}(\mathsf{true}, n, m : x) &\to m : \mathsf{low}(n, x)\\
\mathsf{high}(n, \mathsf{nil}) &\to \mathsf{nil} & \mathsf{if\text{-}high}(\mathsf{false}, n, m : x) &\to m : \mathsf{high}(n, x)\\
\mathsf{high}(n, m : x) &\to \mathsf{if\text{-}high}(m \le n, n, m : x) & \mathsf{if\text{-}high}(\mathsf{true}, n, m : x) &\to \mathsf{high}(n, x)\\
\mathsf{nil} +\!\!+\, y &\to y & 0 \le y &\to \mathsf{true}\\
(n : x) +\!\!+\, y &\to n : (x +\!\!+\, y) & \mathsf{s}(x) \le 0 &\to \mathsf{false}\\
\mathsf{quicksort}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{s}(x) \le \mathsf{s}(y) &\to x \le y\\
\mathsf{quicksort}(n : x) &\to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+ (n : \mathsf{quicksort}(\mathsf{high}(n, x)))\\
\mathsf{quicksort}^\sharp(n : x) &\to \mathsf{quicksort}^\sharp(\mathsf{low}(n, x))\\
\mathsf{quicksort}^\sharp(n : x) &\to \mathsf{quicksort}^\sharp(\mathsf{high}(n, x))
\end{aligned}$$

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DP problem

$$
\begin{aligned}
\mathrm{low}(n, \mathrm{nil}) &\to \mathrm{nil} & \text{if-low}(\mathrm{false}, n, m : x) &\to \mathrm{low}(n, x) \\
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\mathrm{high}(n, m : x) &\to \text{if-high}(m \le n, n, m : x) & \text{if-high}(\mathrm{true}, n, m : x) &\to \mathrm{high}(n, x) \\
\mathrm{nil} \mathbin{+\!\!+} y &\to y & 0 \le y &\to \mathrm{true} \\
(n : x) \mathbin{+\!\!+} y &\to n : (x \mathbin{+\!\!+} y) & \mathrm{s}(x) \le 0 &\to \mathrm{false} \\
\mathrm{quicksort}(\mathrm{nil}) &\to \mathrm{nil} & \mathrm{s}(x) \le \mathrm{s}(y) &\to x \le y
\end{aligned}
$$

$$\mathrm{quicksort}(n : x) \to \mathrm{quicksort}(\mathrm{low}(n, x)) \mathbin{+\!\!+} (n : \mathrm{quicksort}(\mathrm{high}(n, x)))$$

$$\mathrm{quicksort}^{\sharp}(n : x) \to \mathrm{quicksort}^{\sharp}(\mathrm{low}(n, x))$$

$$\mathrm{quicksort}^{\sharp}(n : x) \to \mathrm{quicksort}^{\sharp}(\mathrm{high}(n, x))$$

argument filter $\pi(\mathrm{low}) = \pi(\mathrm{high}) = 2 \quad \pi(\text{if-low}) = \pi(\text{if-high}) = 3$

Example

DP problem

$$
\begin{aligned}
\mathsf{nil} &\to \mathsf{nil} & m : x &\to x \\
m : x &\to \quad m : x & m : x &\to m : \quad x \\
\mathsf{nil} &\to \mathsf{nil} & m : x &\to m : \quad x \\
m : x &\to \quad m : x & m : x &\to x \\
\mathsf{nil} +\!\!+\, y &\to y & 0 \leq y &\to \mathsf{true} \\
(n : x) +\!\!+\, y &\to n : (x +\!\!+\, y) & \mathsf{s}(x) \leq 0 &\to \mathsf{false} \\
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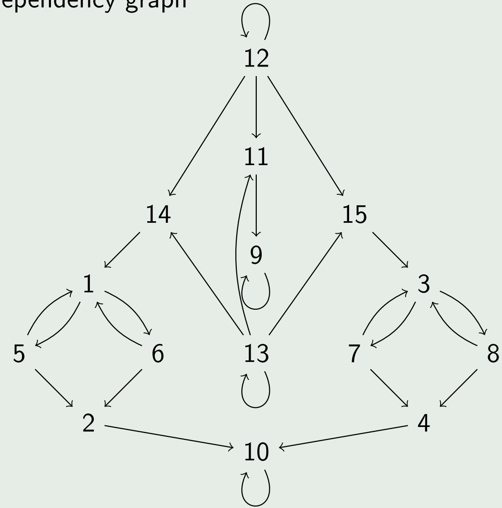
\mathsf{quicksort}(n : x) &\to \mathsf{quicksort}(\quad x \;) +\!\!+ (n : \mathsf{quicksort}(\quad x \;)) \\
\mathsf{quicksort}^{\sharp}(n : x) &\to \mathsf{quicksort}^{\sharp}(\quad x \;) \\
\mathsf{quicksort}^{\sharp}(n : x) &\to \mathsf{quicksort}^{\sharp}(\quad x \;)
\end{aligned}
$$

argument filter $\pi(\mathsf{low}) = \pi(\mathsf{high}) = 2 \quad \pi(\mathsf{if\text{-}low}) = \pi(\mathsf{if\text{-}high}) = 3$

LPO with precedence $\mathsf{quicksort} > +\!\!+ > : \quad \leq > \mathsf{true}, \mathsf{false}$

Example (cont'd)

dependency graph



six SCCs

- $\{9\}$ subterm criterion
- $\{10\}$ subterm criterion
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- $\{13\}$ AF + LPO
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Outline

Definitions (TRS $\mathcal{R}$ over signature $\mathcal{F}$)

skip

- semantics
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$

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  - sets of labels $L_f \subseteq A$ for every $f \in \mathcal{F}$
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- value of terms (for every assignment $\alpha \colon \mathcal{V} \to A$)

$$[\alpha](t) = \begin{cases} \alpha(t) & \text{if } t \in \mathcal{V} \\ f_\mathcal{A}([\alpha](t_1), \dots, [\alpha](t_n)) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

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- labeling of terms (for every assignment $\alpha \colon \mathcal{V} \to A$)

$$\mathsf{lab}_\alpha(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f = \varnothing \\ f_{a}(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

with $a = \ell_f([\alpha]_\mathcal{A}(t_1), \dots, [\alpha]_\mathcal{A}(t_n))$

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with $a = \ell_f([\alpha]_\mathcal{A}(t_1), \dots, [\alpha]_\mathcal{A}(t_n))$

- labeled TRS

$$\mathcal{R}_{\mathsf{lab}} = \{\, \mathsf{lab}_\alpha(\ell) \to \mathsf{lab}_\alpha(r) \mid \ell \to r \in \mathcal{R} \text{ and } \alpha \colon \mathcal{V} \to A \,\}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$$

- algebra $\mathcal{A}$

$$A = \{0, 1\} \qquad \mathsf{a}_{\mathcal{A}} = 0 \qquad \mathsf{b}_{\mathcal{A}} = 1 \qquad \mathsf{f}_{\mathcal{A}}(x, y, z) = 0$$

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$$L_{\mathsf{a}} = L_{\mathsf{b}} = \varnothing \qquad L_{\mathsf{f}} = A \qquad \ell_{\mathsf{f}}(x, y, z) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

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$$\mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x)$$

Example

- TRS $\mathcal{R}$

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$$\mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x)$$

is terminating because it lacks dependency pairs

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$$

- algebra $\mathcal{A}$ is model of $\mathcal{R}$

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is terminating because it lacks dependency pairs

Theorem

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is terminating if

$\exists$ *algebra* $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$

$\exists$ *labeling* ℓ

such that

1. $\mathcal{R} \subseteq =_{\mathcal{A}}$ "$\mathcal{A}$ *is model of* $\mathcal{R}$"
2. $\mathcal{R}_{lab}$ *is terminating*

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Theorem (rephrased)

TRS $\mathcal{R}$, model $\mathcal{A}$, labeling ℓ (semantic labeling is sound)

$$\mathcal{R} \text{ is terminating} \quad \Longleftarrow \quad \mathcal{R}_{lab} \text{ is terminating}$$

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Theorem (rephrased)

TRS $\mathcal{R}$, model $\mathcal{A}$, labeling ℓ (semantic labeling is sound and complete)

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{lab} \text{ is terminating}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

- $\mathsf{a}_{\mathcal{A}} = \mathsf{b}_{\mathcal{A}}$ in every model of $\mathcal{R}$

Outline

skip

Definitions

- well-founded weakly monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ with $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$

skip

Definitions

- well-founded weakly monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ with $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$
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skip

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- $\mathcal{D}\mathrm{ec} = \{\, f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid a, b \in L_f \text{ with } a > b \,\}$

skip

Definitions

- well-founded weakly monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ with $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$
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Theorem

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is terminating if

$\exists$ *well-founded weakly monotone algebra $(\mathcal{A}, >, \geqslant)$*

$\exists$ *weakly monotone labeling ℓ*

such that

1. $\mathcal{R} \subseteq \geqslant_\mathcal{A}$ *"$\mathcal{A}$ is quasi-model of $\mathcal{R}$"*
2. $\mathcal{R}_{lab} \cup \mathcal{D}ec$ *is terminating*

Example

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$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

- well-founded weakly monotone algebra $\mathcal{A}$

$$A = \{0,1,2\} \quad 2 > 0 \quad 1 > 0$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

- well-founded weakly monotone algebra $\mathcal{A}$

$$A = \{0,1,2\} \quad 2 > 0 \quad 1 > 0 \quad \mathsf{a}_{\mathcal{A}} = 1 \quad \mathsf{b}_{\mathcal{A}} = 2 \quad \mathsf{c}_{\mathcal{A}} = \mathsf{f}_{\mathcal{A}}(x,y,z) = 0$$

Example

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$$L_{\mathsf{a}} = L_{\mathsf{b}} = L_{\mathsf{c}} = \varnothing \qquad L_{\mathsf{f}} = \{0,1\} \qquad \ell_{\mathsf{f}}(x,y,z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{if otherwise} \end{cases}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \rightarrow \mathsf{f}(x,x,x) \qquad \mathsf{a} \rightarrow \mathsf{c} \qquad \mathsf{b} \rightarrow \mathsf{c} \qquad \mathsf{f}(x,y,z) \rightarrow \mathsf{c}$$

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- TRS $\mathcal{R}_{\mathsf{lab}}$

$$\mathsf{f}_1(\mathsf{a},\mathsf{b},x) \rightarrow \mathsf{f}_0(x,x,x) \qquad \mathsf{a} \rightarrow \mathsf{c} \qquad \mathsf{b} \rightarrow \mathsf{c} \qquad \begin{array}{l} \mathsf{f}_0(x,y,z) \rightarrow \mathsf{c} \\ \mathsf{f}_1(x,y,z) \rightarrow \mathsf{c} \end{array}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

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Example

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$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

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- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$

$$\begin{array}{llll} \mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x) & \mathsf{a} \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{f}_0(x,y,z) \to \mathsf{c} \\ \mathsf{f}_1(x,y,z) \to \mathsf{f}_0(x,y,z) & & & \mathsf{f}_1(x,y,z) \to \mathsf{c} \end{array}$$

is terminating because dependency graph lacks SCCs

Theorem (rephrased)

TRS $\mathcal{R}$, well-founded quasi-model $\mathcal{A}$, weakly monotone labeling ℓ

$$\mathcal{R} \textit{ is terminating} \quad \Longleftarrow \quad \mathcal{R}_{lab} \cup \mathcal{D}ec \textit{ is terminating}$$

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Theorem (rephrased)

TRS $\mathcal{R}$, well-founded quasi-model $\mathcal{A}$, weakly monotone labeling ℓ

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{lab} \cup \mathcal{D}ec \text{ is terminating}$$

Example

- TRS $\mathcal{R}$

$$\begin{array}{rcl} \lambda(x) \circ y \to \lambda(x \circ (1 \star (y \circ \uparrow))) & \mathsf{id} \circ x \to x & 1 \circ (x \star y) \to x \\ (x \star y) \circ z \to (x \circ z) \star (y \circ z) & 1 \circ \mathsf{id} \to 1 & \uparrow \circ (x \star y) \to y \\ (x \circ y) \circ z \to x \circ (y \circ z) & \uparrow \circ \mathsf{id} \to \uparrow & \end{array}$$

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- well-founded weakly monotone algebra $\mathcal{A}$
  - carrier $A = \mathbb{N}$ with standard order $>$
  - interpretations

$$\begin{array}{ll} \lambda_{\mathcal{A}}(x) = x + 1 & \circ_{\mathcal{A}}(x, y) = x + y \\ \star_{\mathcal{A}}(x, y) = \max(x, y) & 1_{\mathcal{A}} = \uparrow_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}} = 0 \end{array}$$

Example (cont'd)

- weakly monotone labeling ℓ

$$L_\lambda = L_\star = L_1 = L_\uparrow = \varnothing \qquad L_\circ = \mathbb{N} \qquad \ell_\circ(x, y) = x + y$$

Example (cont'd)

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- TRS $\mathcal{R}_{\mathsf{lab}}$ $\qquad \forall\, i,j,k \geqslant 0$

$$\begin{aligned} \lambda(x) \circ_{i+j+1} y &\to \lambda(x \circ_{i+j} (1 \star (y \circ_j \uparrow))) \\ (x \star y) \circ_{\max(i,j)+k} z &\to (x \circ_{i+k} z) \star (y \circ_{j+k} z) \\ (x \circ_{i+j} y) \circ_{i+j+k} z &\to x \circ_{i+j+k} (y \circ_{j+k} z) \\ \mathsf{id} \circ_i x &\to x \qquad\qquad 1 \circ_{\max(i,j)} (x \star y) \to x \\ 1 \circ_0 \mathsf{id} &\to 1 \qquad\qquad \uparrow \circ_{\max(i,j)} (x \star y) \to y \\ \uparrow \circ_0 \mathsf{id} &\to \uparrow \end{aligned}$$

Example (cont'd)

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$$L_\lambda = L_\star = L_1 = L_\uparrow = \varnothing \qquad L_\circ = \mathbb{N} \qquad \ell_\circ(x,y) = x + y$$

- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec} \quad \forall\, i,j,k \geqslant 0 \quad \forall\, l > m \geqslant 0$

$$\begin{aligned}
\lambda(x) \circ_{i+j+1} y &\to \lambda(x \circ_{i+j} (1 \star (y \circ_j \uparrow))) \\
(x \star y) \circ_{\max(i,j)+k} z &\to (x \circ_{i+k} z) \star (y \circ_{j+k} z) \\
(x \circ_{i+j} y) \circ_{i+j+k} z &\to x \circ_{i+j+k} (y \circ_{j+k} z)
\end{aligned}$$

$$\begin{aligned}
\mathsf{id} \circ_i x &\to x & 1 \circ_{\max(i,j)} (x \star y) &\to x \\
1 \circ_0 \mathsf{id} &\to 1 & \uparrow \circ_{\max(i,j)} (x \star y) &\to y \\
\uparrow \circ_0 \mathsf{id} &\to \uparrow & x \circ_l y &\to x \circ_m y
\end{aligned}$$

Example (cont'd)

- weakly monotone labeling ℓ

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- TRS $\mathcal{R}_{\text{lab}} \cup \mathcal{D}\text{ec}$ $\quad \forall\, i,j,k \geqslant 0 \quad \forall\, l > m \geqslant 0$

$$\lambda(x) \circ_{i+j+1} y \to \lambda(x \circ_{i+j} (1 \star (y \circ_j \uparrow)))$$
$$(x \star y) \circ_{\max(i,j)+k} z \to (x \circ_{i+k} z) \star (y \circ_{j+k} z)$$
$$(x \circ_{i+j} y) \circ_{i+j+k} z \to x \circ_{i+j+k} (y \circ_{j+k} z)$$
$$\text{id} \circ_i x \to x \qquad 1 \circ_{\max(i,j)} (x \star y) \to x$$
$$1 \circ_0 \text{id} \to 1 \qquad \uparrow \circ_{\max(i,j)} (x \star y) \to y$$
$$\uparrow \circ_0 \text{id} \to \uparrow \qquad x \circ_l y \to x \circ_m y$$

- LPO with precedence

$$\cdots > \circ_2 > \circ_1 > \circ_0 > \star, \lambda, 1, \uparrow$$

Extensions

- quasi-model condition required for usable rules only (predictive labeling)

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- . . .

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Questions

- how to choose (quasi-)model ?
- how to choose labeling ?

Answers

- ...
- root-labeling

Outline

Definition

skip

- algebra $\mathcal{A}_\mathcal{F}$
  - $A_\mathcal{F} = \mathcal{F}$
  - $f_{\mathcal{A}_\mathcal{F}}(x_1, \ldots, x_n) = f \quad \forall$ n-ary $f \in \mathcal{F}$

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  - $L_f = \mathcal{F}^n$
  - $\ell_f(x_1, \ldots, x_n) = x_1 \cdots x_n$

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Example

$\mathcal{R}$ $\quad \mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$ $\qquad \mathsf{a}_{\mathsf{b}}(\mathsf{b}_{\mathsf{a}}(x)) \to \mathsf{c}_{\mathsf{a}}(x)$ $\quad \mathcal{R}_{\mathsf{rlab}}$

$\alpha(x) = \mathsf{a}$

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$\mathcal{R}_{\mathsf{rlab}}$

$\alpha(x) = \mathsf{b}$

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Example

$\mathcal{R}$ $\quad \mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$

$\alpha(x) = \mathsf{c}$

$$\begin{aligned} \mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{a}(x)) &\to \mathsf{c}_\mathsf{a}(x) \\ \mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{b}(x)) &\to \mathsf{c}_\mathsf{b}(x) \\ \mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{c}(x)) &\to \mathsf{c}_\mathsf{c}(x) \end{aligned}$$

$\mathcal{R}_{\mathsf{rlab}}$

Definition

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Example

$\mathcal{R}$ $\quad \mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$

$$\begin{aligned} \mathsf{a_b}(\mathsf{b_a}(x)) &\to \mathsf{c_a}(x) \\ \mathsf{a_b}(\mathsf{b_b}(x)) &\to \mathsf{c_b}(x) \\ \mathsf{a_b}(\mathsf{b_c}(x)) &\to \mathsf{c_c}(x) \end{aligned}$$

$\mathcal{R}_{\mathsf{rlab}}$

Problem

no model: $\quad \forall\, \alpha \quad [\alpha](\mathsf{a}(\mathsf{b}(x))) = \mathsf{a} \neq \mathsf{c} = [\alpha](\mathsf{c}(x))$

Definitions (TRS $\mathcal{R}$ over signature $\mathcal{F}$)

- root-preserving rules

  $\mathcal{R}_{\mathsf{p}} = \{\, \ell \to r \in \mathcal{R} \mid \mathsf{root}(\ell) = \mathsf{root}(r) \,\}$

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  $\mathcal{FC} = \{\, g(x_1, \ldots, x_{i-1}, \Box, x_i, \ldots, x_n) \mid g \in \mathcal{F}$ has arity n and $1 \leqslant i \leqslant n \,\}$

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Lemma

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Lemma

- *$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})$ is terminating*
- *$\mathcal{A}_{\mathcal{F}}$ is model of $\mathcal{FC}(\mathcal{R})$*

Example

$\mathcal{R}$ $\quad \mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$

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$\mathcal{FC}(\mathcal{R})$ $\quad$ $\mathsf{a}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{a}(\mathsf{c}(x))$ $\quad$ $\mathsf{a}(\square)$

Example

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$\mathcal{FC}(\mathcal{R})$ $\quad$ $\mathsf{a}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{a}(\mathsf{c}(x))$

$\mathsf{b}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{b}(\mathsf{c}(x))$ $\quad$ $\mathsf{b}(\square)$

Example

$\mathcal{R}$ $\quad$ $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$

$\mathcal{FC}(\mathcal{R})$

$$\mathsf{a}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{a}(\mathsf{c}(x))$$
$$\mathsf{b}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{b}(\mathsf{c}(x))$$
$$\mathsf{c}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{c}(\mathsf{c}(x)) \qquad \mathsf{c}(\Box)$$

Example

$\mathcal{R}$

$$a(b(x)) \to c(x)$$

$\mathcal{FC}(\mathcal{R})$

$$a(a(b(x))) \to a(c(x))$$
$$b(a(b(x))) \to b(c(x))$$
$$c(a(b(x))) \to c(c(x))$$

$\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$

$$a_a(a_b(b_a(x))) \to a_c(c_a(x))$$
$$a_a(a_b(b_b(x))) \to a_c(c_b(x))$$
$$a_a(a_b(b_c(x))) \to a_c(c_c(x))$$
$$b_a(a_b(b_a(x))) \to b_c(c_a(x))$$
$$b_a(a_b(b_b(x))) \to b_c(c_b(x))$$
$$b_a(a_b(b_c(x))) \to b_c(c_c(x))$$
$$c_a(a_b(b_a(x))) \to c_c(c_a(x))$$
$$c_a(a_b(b_b(x))) \to c_c(c_b(x))$$
$$c_a(a_b(b_c(x))) \to c_c(c_c(x))$$

Corollary

$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})_{rlab}$ is terminating

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Example (1)

- $\mathcal{R}$ $\quad$ $\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$

Corollary

$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})_{rlab}$ is terminating

Example (1)

- $\mathcal{R}$ $\qquad \mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$

Corollary

$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})_{rlab}$ is terminating

Example (1)

- $\mathcal{R}$ $\quad \mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ $\quad \begin{aligned} \mathsf{f}_{\mathsf{aba}}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}_{\mathsf{aaa}}(x, x, x) \\ \mathsf{f}_{\mathsf{abb}}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}_{\mathsf{bbb}}(x, x, x) \\ \mathsf{f}_{\mathsf{abf}}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}_{\mathsf{fff}}(x, x, x) \end{aligned}$

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Example (1)

- $\mathcal{R}$ $\quad \mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$
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- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ is terminating because it lacks dependency pairs

Corollary

$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})_{rlab}$ is terminating

Example (2)

- $\mathcal{R}$ $\quad$ $\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$

Corollary

$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})_{rlab}$ is terminating

Example (2)

- $\mathcal{R}$ $\quad$ $\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$
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Corollary

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Example (2)

- $\mathcal{R}$ $\quad$ $\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$
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  $\mathsf{f}_\mathsf{f}(\mathsf{f}_\mathsf{g}(x)) \to \mathsf{f}_\mathsf{g}(\mathsf{g}_\mathsf{f}(\mathsf{f}_\mathsf{g}(x)))$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ is polynomially terminating

Outline

- Matrix Interpretations
- Dependency Pairs
- Semantic Labeling
- Derivational Complexity
  - Polynomial Interpretations
  - Triangular Matrix Interpretations
- Further Reading

Definitions

- derivation length $\mathrm{dl}(t) = \max\{\, n \mid \exists\, u\colon t \to^n u \,\}$

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Example

rewrite rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x))$

term	derivation length
a(b(b(b(c))))	

Definitions

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Example

rewrite rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x))$

term	derivation length
a(b(b(b(c))))	3
a(a(a(b(b(b(c))))))	

Definitions

- derivation length $\mathrm{dl}(t) = \max\{\, n \mid \exists\, u\colon\ t \to^n u \,\}$

Example

rewrite rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x))$

term	derivation length
$\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))$	3
$\mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))))$	9
$\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))$	

Definitions

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$\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))$	mn

Definitions

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$\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))$	mn

polynomial interpretation

$$\mathsf{a}_{\mathbb{N}}(x) = 2x \qquad \mathsf{b}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{c}_{\mathbb{N}} = 0$$

Definitions

- derivation length $\quad \mathrm{dl}(t) = \max \{ n \mid \exists\, u \colon t \to^n u \}$

Example

rewrite rule $\quad \mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x))$

term	derivation length	value
a(b(b(b(c))))	3	6
a(a(a(b(b(b(c))))))	9	24
$\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))$	mn	$2^m n$

polynomial interpretation

$$\mathsf{a}_{\mathbb{N}}(x) = 2x \qquad \mathsf{b}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{c}_{\mathbb{N}} = 0$$

Definitions

- derivation length $\mathrm{dl}(t) = \max \{ n \mid \exists\, u \colon t \to^n u \}$
- derivational complexity $\mathrm{dc}(n) = \max \{ \mathrm{dl}(t) \mid |t| = n \}$

Example

rewrite rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x))$

term	derivation length	value
$\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))$	3	6
$\mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))))$	9	24
$\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))$	mn	$2^m n$

polynomial interpretation

$$\mathsf{a}_{\mathbb{N}}(x) = 2x \qquad \mathsf{b}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{c}_{\mathbb{N}} = 0$$

Theorem

interpretation in $\mathbb{N}$	*bound on derivational complexity*
polynomial	*double-exponential*

Theorem

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Example

rewrite system

$$x + 0 \to x \qquad \mathsf{d}(0) \to 0 \qquad \mathsf{q}(0) \to 0$$
$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad \mathsf{d}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \qquad \mathsf{q}(\mathsf{s}(x)) \to \mathsf{q}(x) + \mathsf{s}(\mathsf{d}(x))$$

interpretations

$$0_{\mathbb{N}} = 2 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad +_{\mathbb{N}}(x, y) = x + 2y \quad \mathsf{d}_{\mathbb{N}}(x) = 3x \quad \mathsf{q}_{\mathbb{N}}(x) = x^3$$

Theorem

	interpretation in $\mathbb{N}$	*bound on derivational complexity*
	polynomial	*double-exponential*
$a_1 x_1 + \cdots + a_n x_n + b$	*linear*	*exponential*

Example

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Theorem

	interpretation in $\mathbb{N}$	*bound on derivational complexity*
	polynomial	*double-exponential*
$a_1x_1 + \cdots + a_nx_n + b$	*linear*	*exponential*
$x_1 + \cdots + x_n + b$	*strongly linear*	*linear*

Example

rewrite system

$$x + 0 \to x \qquad \mathsf{d}(0) \to 0 \qquad \mathsf{q}(0) \to 0$$

$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad \mathsf{d}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \qquad \mathsf{q}(\mathsf{s}(x)) \to \mathsf{q}(x) + \mathsf{s}(\mathsf{d}(x))$$

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Outline

Definition

triangular matrix interpretation $\mathcal{M}$ over $\mathbb{N}$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$
- interpretations (for every n-ary f)

$$f_{\mathcal{M}}(\vec{x}_1, \dots, \vec{x}_n) = M_1\vec{x}_1 + \cdots + M_n\vec{x}_n + \vec{f}$$

  with

  - matrices $M_1, \dots, M_n \in \mathbb{N}^{d \times d}$ with

$$(M_i)_{1,1} \geqslant 1 \qquad (M_i)_{j,k} = 0 \text{ for all } j > k \qquad (M_i)_{j,j} \leqslant 1 \text{ for all } j$$

    for all $1 \leqslant i \leqslant n$
  - vector $\vec{f} \in \mathbb{N}^d$

- $(x_1, \dots, x_d)^\mathsf{T} > (y_1, \dots, y_d)^\mathsf{T} \iff x_1 > y_1 \wedge \bigwedge_{i=2}^{d} x_i \geqslant y_i$

Theorem

if $\mathcal{R} \subseteq >_{\mathcal{M}}$ *for triangular matrix interpretation* $\mathcal{M}$ *of dimension* d *then*

$$\mathrm{dc}(n) = \mathcal{O}(n^d)$$

Theorem

if $\mathcal{R} \subseteq >_{\mathcal{M}}$ *for triangular matrix interpretation* $\mathcal{M}$ *of dimension* d *then*

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Example

rewrite rules

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x)) \qquad \mathsf{c}(\mathsf{a}(x)) \to \mathsf{b}(\mathsf{c}(x)) \qquad \mathsf{c}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{c}(x))$$

triangular matrix interpretation

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathsf{c}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Outline

- Matrix Interpretations
- Dependency Pairs
- Semantic Labeling
- Derivational Complexity
- Further Reading

- Matrix Interpretations for Proving Termination of Term Rewriting
  Jörg Endrullis, Johannes Waldmann and Hans Zantema
  JAR 40(2,3), pp. 195 – 220, 2008

- Mechanizing and Improving Dependency Pairs
  Jürgen Giesl, René Thiemann, Peter Schneider-Kamp and Stephan Falke
  JAR 37(3), pp. 155 – 203, 2006

- Tyrolean Termination Tool: Techniques and Features
  Nao Hirokawa and Aart Middeldorp
  I&C 205(4), pp. 474 – 511, 2007

- Termination of Term Rewriting by Semantic Labelling
  Hans Zantema
  FI 24(1,2), pp. 89 – 105, 1995

- Root-Labeling
  Christian Sternagel and Aart Middeldorp
  Proc. 19th RTA, LNCS 5117, pp. 336 – 350, 2008

Complexity Analysis of Term Rewriting Based on Matrix and Context Dependent Interpretations
Georg Moser, Andreas Schnabl and Johannes Waldmann
Proc. 28th FSTTCS, LIPI 2, pp. 304 – 315, 2008

Termination Tools

- AProVE
- CiME
- Jambox
- Matchbox
- MuTerm
- TTT2
- VMTL
- ...

Complexity Tools

- CaT
- TCT

Definition

- $\text{tcap}_{\mathcal{R}}(t) = f(\text{tcap}_{\mathcal{R}}(t_1), \ldots, \text{tcap}_{\mathcal{R}}(t_n))$

  if $t = f(t_1, \ldots, t_n)$ and $f(\text{tcap}_{\mathcal{R}}(t_1), \ldots, \text{tcap}_{\mathcal{R}}(t_n))$ does not unify with any left-hand side of $\mathcal{R}$

- $\text{tcap}_{\mathcal{R}}(t)$ is fresh variable

  otherwise

return

Definition

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  if $t = f(t_1, \ldots, t_n)$ and $f(\text{tcap}_{\mathcal{R}}(t_1), \ldots, \text{tcap}_{\mathcal{R}}(t_n))$ does not unify with any left-hand side of $\mathcal{R}$

- $\text{tcap}_{\mathcal{R}}(t)$ is fresh variable

  otherwise

Definition (Modern Approximation)

$$s \to t \xrightarrow{\text{m}} u \to v \quad \Longleftrightarrow \quad \begin{cases} \text{tcap}_{\mathcal{R}}(t) \text{ and } u \text{ are unifiable} \\ t \text{ and } \text{tcap}_{\mathcal{R}^{-1}}(u) \text{ are unifiable} \end{cases}$$

return