Introduction to Term Rewriting

lecture 10

Aart Middeldorp and Femke van Raamsdonk

Institute of Computer Science
University of Innsbruck

Department of Computer Science
VU Amsterdam

Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

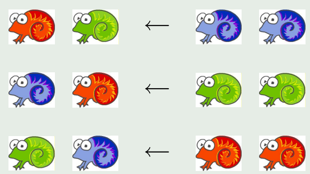
exam, advanced topics

Outline

- Matrix Interpretations
- Dependency Pairs
- Semantic Labeling
- Derivational Complexity
- Further Reading

Example

SRS



is terminating ! (RTA open problem # 104)

Definition

algebra $\mathcal{M}$ with well-founded order $>$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$
- interpretations (for every n-ary f)

$$f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = M_1\vec{x}_1 + \cdots + M_n\vec{x}_n + \vec{f}$$

  with

  - matrices $M_1, \ldots, M_n \in \mathbb{N}^{d\times d}$ with $(M_i)_{1,1} \geqslant 1$ for all $1 \leqslant i \leqslant n$
  - vector $\vec{f} \in \mathbb{N}^d$

- $(x_1, \ldots, x_d)^{\mathsf{T}} > (y_1, \ldots, y_d)^{\mathsf{T}} \iff x_1 > y_1 \wedge \bigwedge_{i=2}^{d} x_i \geqslant y_i$

Lemma

$(\mathcal{M}, >)$ is well-founded monotone algebra

Example (1)

rewrite rules

$$\begin{aligned} \mathsf{a}(\mathsf{a}(x)) &\to \mathsf{b}(\mathsf{c}(x)) \\ \mathsf{b}(\mathsf{b}(x)) &\to \mathsf{a}(\mathsf{c}(x)) \\ \mathsf{c}(\mathsf{c}(x)) &\to \mathsf{a}(\mathsf{b}(x)) \end{aligned}$$

matrix interpretation

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathsf{c}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\forall \vec{x} \quad \begin{aligned} \mathsf{a}_{\mathcal{M}}(\mathsf{a}_{\mathcal{M}}(\vec{x})) &> \mathsf{b}_{\mathcal{M}}(\mathsf{c}_{\mathcal{M}}(\vec{x})) \\ \mathsf{b}_{\mathcal{M}}(\mathsf{b}_{\mathcal{M}}(\vec{x})) &> \mathsf{a}_{\mathcal{M}}(\mathsf{c}_{\mathcal{M}}(\vec{x})) \\ \mathsf{c}_{\mathcal{M}}(\mathsf{c}_{\mathcal{M}}(\vec{x})) &> \mathsf{a}_{\mathcal{M}}(\mathsf{b}_{\mathcal{M}}(\vec{x})) \end{aligned}$$

Example (2)

rewrite rules

$$\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{c}) \qquad \mathsf{h}(\mathsf{c}) \to \mathsf{h}(\mathsf{a})$$

matrix interpretation

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{a}_{\mathcal{M}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{b}_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathsf{h}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{c}_{\mathcal{M}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Matrix Interpretations

Example (3)

rewrite rules

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x) \qquad \mathsf{i}(\mathsf{a}_1, x, \mathsf{b}_1) \to \mathsf{i}(x, \mathsf{a}_2, \mathsf{b}_1)$$

$$\mathsf{g}(\mathsf{h}(x)) \to \mathsf{f}(\mathsf{h}(\mathsf{f}(x))) \qquad \mathsf{i}(x, \mathsf{a}_3, \mathsf{b}_1) \to \mathsf{i}(\mathsf{a}_4, x, \mathsf{b}_1)$$

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) \to \mathsf{i}(x, x, x) \qquad \mathsf{i}(x, x, \mathsf{b}_1) \to \mathsf{i}(\mathsf{g}(x), \mathsf{b}_2, \mathsf{b}_2)$$

$$\mathsf{i}(\mathsf{g}(x), \mathsf{b}_3, \mathsf{b}_3) \to \mathsf{i}(x, x, \mathsf{b}_4)$$

matrix interpretation

$$\mathsf{f}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathsf{a}_{1\mathcal{A}} = \mathsf{a}_{3\mathcal{A}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathsf{b}_{1\mathcal{A}} = \begin{pmatrix} 13 \\ 0 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{a}_{2\mathcal{A}} = \mathsf{a}_{4\mathcal{A}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \mathsf{b}_{2\mathcal{A}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\mathsf{h}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{3\mathcal{A}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{4\mathcal{A}} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$\mathsf{i}_{\mathcal{A}}(\vec{x}, \vec{y}, \vec{z}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Outline

Definition

DP problem is pair of finite TRSs $(\mathcal{P}, \mathcal{R})$ such that root symbols of rules in $\mathcal{P}$

- do not occur in $\mathcal{R}$
- do not occur in proper subterms of left- and right-hand sides of rules in $\mathcal{P}$

Example

$(\mathrm{DP}(\mathcal{R}), \mathcal{R})$ is DP problem for every TRS $\mathcal{R}$

Definition

DP problem $(\mathcal{P}, \mathcal{R})$ is finite if $\neg\,\exists$ infinite sequence

$$t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathcal{P}]{\epsilon} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathcal{P}]{\epsilon} \cdots$$

such that all terms t_1, t_2, ... are terminating with respect to $\mathcal{R}$

Theorem

finite TRS $\mathcal{R}$ is terminating $\iff$ *DP problem $(DP(\mathcal{R}), \mathcal{R})$ is finite*

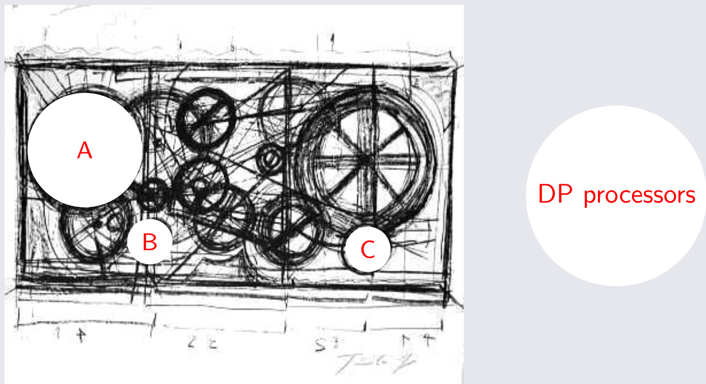
Definitions

- DP processor is function from DP problems to sets of DP problems
- DP processor Φ is sound if DP problem $(\mathcal{P}, \mathcal{R})$ is finite whenever all DP problems in $\Phi(\mathcal{P}, \mathcal{R})$ are finite
- DP processor Φ is complete if DP problem $(\mathcal{P}, \mathcal{R})$ is not finite whenever at least one DP problem in $\Phi(\mathcal{P}, \mathcal{R})$ is not finite

DP Processors

dependency graph, increasing interpretations, instantiation, loop detection, match-bounds, narrowing, reduction pairs, rewriting, root-labeling, rule removal, size-change principle, semantic labeling, string reversal, subterm criterion, uncurrying, usable rules, . . .

Termination Tools



Outline

Definition

dependency graph $\mathrm{DG}(\mathcal{P},\mathcal{R})$ of DP problem $(\mathcal{P},\mathcal{R})$ consists of

- nodes: rules in $\mathcal{P}$
- arrows: $s \to t \longrightarrow u \to v$ if $\exists$ substitutions σ, τ such that $t\sigma \xrightarrow[\mathcal{R}]{*} u\tau$

Definition (Dependency Graph Processor)

$(\mathcal{P},\mathcal{R}) \mapsto \{(\mathcal{S},\mathcal{R}) \mid \mathcal{S}$ is strongly connected component in $\mathrm{DG}(\mathcal{P},\mathcal{R})\}$

Theorem

dependency graph processor is sound and complete

Remark

dependency graph is not computable but good over-approximations exist

Trivial Approximation

$s \to t \xrightarrow{\text{t}} u \to v \quad \iff \quad \text{root}(t) = \text{root}(u)$

Better Approximations

are based on

- unification
- tree automata techniques

Outline

Definition

DP problem $(\mathcal{P}, \mathcal{R})$ is finite if $\neg\,\exists$ infinite sequence

$$t_1 \gtrsim t_2 > t_3 \gtrsim t_4 > \cdots$$

such that all terms t_1, t_2, ... are terminating with respect to $\mathcal{R}$

Question

how to prove finiteness of DP problem $(\mathcal{P}, \mathcal{R})$?

Answers

- use pair of orders $\gtrsim$ and $>$ such that $\mathcal{R} \subseteq \gtrsim$ and $\mathcal{P} \subseteq >$
- use minimality together with subterm relation
- ...

Definition

reduction pair $(>, \gtrsim)$ consists of well-founded order $>$ and preorder $\gtrsim$ such that

1. $>$ is closed under substitutions
2. $\gtrsim$ is closed under contexts and substitutions
3. $> \cdot \gtrsim \subseteq >$ or $\gtrsim \cdot > \subseteq >$

Definition (Reduction Pair Processor)

reduction pair $(>, \gtrsim)$

$$(\mathcal{P}, \mathcal{R}) \mapsto \begin{cases} \{(\mathcal{P} \setminus >, \mathcal{R})\} & \text{if } \mathcal{P} \subseteq > \cup \gtrsim \text{ and } \mathcal{R} \subseteq \gtrsim \\ \{(\mathcal{P}, \mathcal{R})\} & \text{otherwise} \end{cases}$$

Theorem

reduction pair processor is sound and complete

Definition

monotonic reduction pair is reduction pair $(>, \gtrsim)$ with $>$ closed under contexts

Definition (Rule Removal Processor)

monotonic reduction pair $(>, \gtrsim)$

$$(\mathcal{P}, \mathcal{R}) \mapsto \begin{cases} \{(\mathcal{P} \setminus >, \mathcal{R} \setminus >)\} & \text{if } \mathcal{P} \subseteq > \cup \gtrsim \text{ and } \mathcal{R} \subseteq \gtrsim \cup > \\ \{(\mathcal{P}, \mathcal{R})\} & \text{otherwise} \end{cases}$$

Theorem

rule removal processor is sound and complete

Example

rewrite rules

$$0 + y \geqslant_{\mathbb{N}} y$$
$$\mathsf{s}(x) + y >_{\mathbb{N}} \mathsf{s}(x + y)$$

dependency pairs

$$\mathsf{s}(x) +^\sharp y >_{\mathbb{N}} x +^\sharp y$$

polynomial interpretations

$$0_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad +_{\mathbb{N}}(x, y) = +^\sharp_{\mathbb{N}}(x, y) = 2x + y$$

reduction pair processor vs rule removal processor

Outline

Remark

traditional simplification orders like LPO and KBO give rise to monotonic reduction pairs

Definition

- argument filter is mapping π such that for every n-ary function symbol f

$$\pi(f) \in \{1, \dots, n\} \quad \text{or} \quad \pi(f) = [i_1, \dots, i_m] \text{ with } 1 \leqslant i_1 < \cdots < i_m \leqslant n$$

- extension to terms:

$$\pi(t) = \begin{cases} t & \text{if } t \text{ is variable} \\ \pi(t_i) & \text{if } t = f(t_1, \dots, t_n) \text{ and } \pi(f) = i \\ f(\pi(t_{i_1}), \dots, \pi(t_{i_m})) & \text{if } t = f(t_1, \dots, t_n) \text{ and } \pi(f) = [i_1, \dots, i_m] \end{cases}$$

Notation

- $s >_\pi t \iff \pi(s) > \pi(t)$
- $s \gtrsim_\pi t \iff \pi(s) \gtrsim \pi(t)$

Lemma

$(>_\pi, \gtrsim_\pi)$ is reduction pair for every reduction pair $(>, \gtrsim)$ and argument filter π

Example

DP problem

$$0 - y \to 0 \qquad 0 \div \mathsf{s}(y) \to 0$$
$$x - 0 \to x \qquad \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y \qquad \mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to (x - y) \div^\sharp \mathsf{s}(y)$$

argument filter $\pi(-) = 1$ LPO with precedence $\div > \mathsf{s}$

Outline

Definition

DP problem $(\mathcal{P}, \mathcal{R})$ is finite if $\neg\,\exists$ infinite sequence

$$t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[\mathcal{P}]{\epsilon} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[\mathcal{P}]{\epsilon} \cdots$$

such that all terms t_1, t_2, ... are terminating with respect to $\mathcal{R}$

Question

how to prove finiteness of DP problem $(\mathcal{P}, \mathcal{R})$?

Answers

- use pair of orders
- use minimality together with subterm relation
- ...

Idea

project each dependency pair symbol in $\mathcal{P}$ to fixed argument position

$$\pi(t_1) \xrightarrow[\mathcal{R}]{*} \pi(t_2) \quad ? \quad \pi(t_3) \xrightarrow[\mathcal{R}]{*} \pi(t_4) \quad ? \quad \cdots$$

Observation

$\pi(t_1)$ is terminating with respect to $\xrightarrow[\mathcal{R}]{} \cup \rhd$

Definition

- simple projection for DP problem $(\mathcal{P}, \mathcal{R})$ is mapping π that assigns to every dependency pair symbol $f^\sharp$ in $\mathcal{P}$ one of its argument positions
- extension to terms in $\mathcal{T}^\sharp$: $\pi(f^\sharp(t_1, \ldots, t_n)) = t_{\pi(f^\sharp)}$

Definition (Subterm Criterion Processor)

simple projection π

$$(\mathcal{P},\mathcal{R}) \mapsto \begin{cases} \{(\{\ell \to r \in \mathcal{P} \mid \pi(\ell) = \pi(r)\}, \mathcal{R})\} & \text{if } \pi(\mathcal{P}) \subseteq \trianglerighteq \\ \{(\mathcal{P},\mathcal{R})\} & \text{otherwise} \end{cases}$$

Theorem

subterm criterion processor is sound and complete

Example

rewrite rules

$$\begin{aligned} \mathsf{ack}(0, y) &\to \mathsf{s}(y) \\ \mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

dependency pairs

$$\begin{aligned} \mathsf{ack}^\sharp(\mathsf{s}(x), 0) &\to \mathsf{ack}^\sharp(x, \mathsf{s}(0)) \\ \mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}^\sharp(x, \mathsf{ack}(\mathsf{s}(x), y)) \\ \mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}^\sharp(\mathsf{s}(x), y) \end{aligned}$$

simple projection

$$\pi(\mathsf{ack}^\sharp) = 1$$

Example

rewrite rules

$$\mathsf{ack}(0, y) \to \mathsf{s}(y)$$
$$\mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$
$$\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

dependency pairs

$$\mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}^\sharp(\mathsf{s}(x), y)$$

simple projection

$$\pi(\mathsf{ack}^\sharp) = 2$$

Example

rewrite rules

$$\mathsf{low}(n, \mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{if\text{-}low}(\mathsf{false}, n, m : x) \to \mathsf{low}(n, x)$$
$$\mathsf{low}(n, m : x) \to \mathsf{if\text{-}low}(m \leq n, n, m : x) \qquad \mathsf{if\text{-}low}(\mathsf{true}, n, m : x) \to m : \mathsf{low}(n, x)$$
$$\mathsf{high}(n, \mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{if\text{-}high}(\mathsf{false}, n, m : x) \to m : \mathsf{high}(n, x)$$
$$\mathsf{high}(n, m : x) \to \mathsf{if\text{-}high}(m \leq n, n, m : x) \qquad \mathsf{if\text{-}high}(\mathsf{true}, n, m : x) \to \mathsf{high}(n, x)$$
$$\mathsf{nil} +\!\!+ y \to y \qquad 0 \leq y \to \mathsf{true}$$
$$(n : x) +\!\!+ y \to n : (x +\!\!+ y) \qquad \mathsf{s}(x) \leq 0 \to \mathsf{false}$$
$$\mathsf{quicksort}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{s}(x) \leq \mathsf{s}(y) \to x \leq y$$
$$\mathsf{quicksort}(n : x) \to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+ (n : \mathsf{quicksort}(\mathsf{high}(n, x)))$$

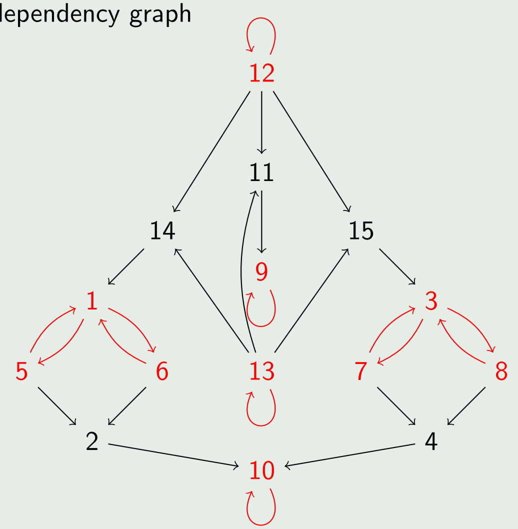
Example (cont'd)

dependency pairs

1	$\mathsf{low}^\sharp(n, m : x) \to \mathsf{if\text{-}low}^\sharp(m \leq n, n, m : x)$	5	$\mathsf{if\text{-}low}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
2	$\mathsf{low}^\sharp(n, m : x) \to m \leq^\sharp n$	6	$\mathsf{if\text{-}low}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{low}^\sharp(n, x)$
3	$\mathsf{high}^\sharp(n, m : x) \to \mathsf{if\text{-}high}^\sharp(m \leq n, n, m : x)$	7	$\mathsf{if\text{-}high}^\sharp(\mathsf{false}, n, m : x) \to \mathsf{high}^\sharp(n, x)$
4	$\mathsf{high}^\sharp(n, m : x) \to m \leq^\sharp n$	8	$\mathsf{if\text{-}high}^\sharp(\mathsf{true}, n, m : x) \to \mathsf{high}^\sharp(n, x)$
9	$(n : x) +\!\!+^\sharp y \to x +\!\!+^\sharp y$	10	$\mathsf{s}(x) \leq^\sharp \mathsf{s}(y) \to x \leq^\sharp y$
11	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}(\mathsf{low}(n, x)) +\!\!+^\sharp (n : \mathsf{quicksort}(\mathsf{high}(n, x)))$		
12	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}^\sharp(\mathsf{low}(n, x))$	14	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{low}^\sharp(n, x)$
13	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{quicksort}^\sharp(\mathsf{high}(n, x))$	15	$\mathsf{quicksort}^\sharp(n : x) \to \mathsf{high}^\sharp(n, x)$

Example (cont'd)

dependency graph



six SCCs

- $\{9\}$ subterm criterion
- $\{10\}$ subterm criterion
- $\{12\}$ AF + LPO
- $\{13\}$ AF + LPO
- $\{1, 5, 6\}$ subterm criterion
- $\{3, 7, 8\}$ subterm criterion

Outline

- Matrix Interpretations
- Dependency Pairs
- Semantic Labeling
  - Model Version
  - Quasi-Model Version
  - Root-Labeling
- Derivational Complexity
- Further Reading

Definitions (TRS $\mathcal{R}$ over signature $\mathcal{F}$)

- semantics
  - $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$
- labeling
  - sets of labels $L_f \subseteq A$ for every $f \in \mathcal{F}$
  - labeling functions $\ell_f \colon A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \varnothing$
- labeling of terms (for every assignment $\alpha \colon \mathcal{V} \to A$)

$$\mathsf{lab}_\alpha(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f = \varnothing \\ f_a(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \dots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

  with $a = \ell_f([\alpha]_\mathcal{A}(t_1), \dots, [\alpha]_\mathcal{A}(t_n))$
- labeled TRS

$$\mathcal{R}_{\mathsf{lab}} = \{\, \mathsf{lab}_\alpha(\ell) \to \mathsf{lab}_\alpha(r) \mid \ell \to r \in \mathcal{R} \text{ and } \alpha \colon \mathcal{V} \to A \,\}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

- algebra $\mathcal{A}$ is model of $\mathcal{R}$

$$A = \{0,1\} \qquad \mathsf{a}_\mathcal{A} = 0 \qquad \mathsf{b}_\mathcal{A} = 1 \qquad \mathsf{f}_\mathcal{A}(x,y,z) = 0$$

- labeling ℓ

$$L_\mathsf{a} = L_\mathsf{b} = \varnothing \qquad L_\mathsf{f} = A \qquad \ell_\mathsf{f}(x,y,z) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

- TRS $\mathcal{R}_\mathsf{lab}$

$$\mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x)$$

is terminating because it lacks dependency pairs

Theorem

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is terminating if

$\exists$ *algebra* $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$

$\exists$ *labeling* ℓ

such that

1. $\mathcal{R} \subseteq =_\mathcal{A}$ *"$\mathcal{A}$ is model of $\mathcal{R}$"*
2. $\mathcal{R}_{lab}$ *is terminating*

Theorem (rephrased)

TRS $\mathcal{R}$, model $\mathcal{A}$, labeling ℓ (semantic labeling is sound and complete)

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{lab} \text{ is terminating}$$

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x,y,z) \to \mathsf{c}$$

- well-founded weakly monotone algebra $\mathcal{A}$

$$A = \{0,1,2\} \quad 2 > 0 \quad 1 > 0 \quad \mathsf{a}_{\mathcal{A}} = 1 \quad \mathsf{b}_{\mathcal{A}} = 2 \quad \mathsf{c}_{\mathcal{A}} = \mathsf{f}_{\mathcal{A}}(x,y,z) = 0$$

- weakly monotone labeling ℓ

$$L_{\mathsf{a}} = L_{\mathsf{b}} = L_{\mathsf{c}} = \varnothing \qquad L_{\mathsf{f}} = \{0,1\} \qquad \ell_{\mathsf{f}}(x,y,z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{if otherwise} \end{cases}$$

- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$

$$\begin{array}{llll} \mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x) & \mathsf{a} \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{f}_0(x,y,z) \to \mathsf{c} \\ \mathsf{f}_1(x,y,z) \to \mathsf{f}_0(x,y,z) & & & \mathsf{f}_1(x,y,z) \to \mathsf{c} \end{array}$$

is terminating because dependency graph lacks SCCs

Outline

Definitions

- well-founded weakly monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ with $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$
- weakly monotone labeling functions $\ell_f \colon A^n \to L_f$ for every n-ary $f \in \mathcal{F}$ with $L_f \neq \varnothing$
- $\mathcal{D}\mathrm{ec} = \{\, f_a(x_1, \dots, x_n) \to f_b(x_1, \dots, x_n) \mid a, b \in L_f \text{ with } a > b \,\}$

Theorem

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is terminating if
 $\exists$ *well-founded weakly monotone algebra $(\mathcal{A}, >, \geqslant)$*
 $\exists$ *weakly monotone labeling ℓ*
such that

1. $\mathcal{R} \subseteq \geqslant_{\mathcal{A}}$ *"$\mathcal{A}$ is quasi-model of $\mathcal{R}$"*
2. *$\mathcal{R}_{lab} \cup \mathcal{D}ec$ is terminating*

Example

- TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{f}(x, y, z) \to \mathsf{c}$$

- well-founded weakly monotone algebra $\mathcal{A}$

$$A = \{0, 1, 2\} \quad 2 > 0 \quad 1 > 0 \quad \mathsf{a}_{\mathcal{A}} = 1 \quad \mathsf{b}_{\mathcal{A}} = 2 \quad \mathsf{c}_{\mathcal{A}} = \mathsf{f}_{\mathcal{A}}(x, y, z) = 0$$

- weakly monotone labeling ℓ

$$L_{\mathsf{a}} = L_{\mathsf{b}} = L_{\mathsf{c}} = \varnothing \qquad L_{\mathsf{f}} = \{0, 1\} \qquad \ell_{\mathsf{f}}(x, y, z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{if otherwise} \end{cases}$$

- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathrm{ec}$

$$\begin{array}{llll} \mathsf{f}_1(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}_0(x, x, x) & \mathsf{a} \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{f}_0(x, y, z) \to \mathsf{c} \\ \mathsf{f}_1(x, y, z) \to \mathsf{f}_0(x, y, z) & & & \mathsf{f}_1(x, y, z) \to \mathsf{c} \end{array}$$

  is terminating because dependency graph lacks SCCs

Theorem (rephrased)

TRS $\mathcal{R}$, well-founded quasi-model $\mathcal{A}$, weakly monotone labeling ℓ

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{lab} \cup \mathcal{D}ec \text{ is terminating}$$

Example

- TRS $\mathcal{R}$

$$\begin{array}{lll} \lambda(x) \circ y \to \lambda(x \circ (1 \star (y \circ \uparrow))) & \mathsf{id} \circ x \to x & 1 \circ (x \star y) \to x \\ (x \star y) \circ z \to (x \circ z) \star (y \circ z) & 1 \circ \mathsf{id} \to 1 & \uparrow \circ (x \star y) \to y \\ (x \circ y) \circ z \to x \circ (y \circ z) & \uparrow \circ \mathsf{id} \to \uparrow & \end{array}$$

- well-founded weakly monotone algebra $\mathcal{A}$
  - carrier $A = \mathbb{N}$ with standard order $>$
  - interpretations

$$\begin{array}{ll} \lambda_{\mathcal{A}}(x) = x + 1 & \circ_{\mathcal{A}}(x, y) = x + y \\ \star_{\mathcal{A}}(x, y) = \max(x, y) & 1_{\mathcal{A}} = \uparrow_{\mathcal{A}} = \mathsf{id}_{\mathcal{A}} = 0 \end{array}$$

Example (cont'd)

- weakly monotone labeling ℓ

$$L_\lambda = L_\star = L_1 = L_\uparrow = \varnothing \qquad L_\circ = \mathbb{N} \qquad \ell_\circ(x, y) = x + y$$

- TRS $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$ $\quad \forall\, i, j, k \geqslant 0 \quad \forall\, l > m \geqslant 0$

$$\begin{array}{c} \lambda(x) \circ_{i+j+1} y \to \lambda(x \circ_{i+j} (1 \star (y \circ_j \uparrow))) \\ (x \star y) \circ_{\max(i,j)+k} z \to (x \circ_{i+k} z) \star (y \circ_{j+k} z) \\ (x \circ_{i+j} y) \circ_{i+j+k} z \to x \circ_{i+j+k} (y \circ_{j+k} z) \end{array}$$

$$\begin{array}{ll} \mathsf{id} \circ_i x \to x & 1 \circ_{\max(i,j)} (x \star y) \to x \\ 1 \circ_0 \mathsf{id} \to 1 & \uparrow \circ_{\max(i,j)} (x \star y) \to y \\ \uparrow \circ_0 \mathsf{id} \to \uparrow & x \circ_l y \to x \circ_m y \end{array}$$

- LPO with precedence

$$\cdots > \circ_2 > \circ_1 > \circ_0 > \star, \lambda, 1, \uparrow$$

Extensions

- quasi-model condition required for usable rules only (predictive labeling)
- incorporation of argument filters
- DP processors based on semantic labeling

Questions

- how to choose (quasi-)model ?
- how to choose labeling ?

Answers

- ...
- root-labeling

Outline

Definition

- algebra $\mathcal{A}_{\mathcal{F}}$
  - $A_{\mathcal{F}} = \mathcal{F}$
  - $f_{\mathcal{A}_{\mathcal{F}}}(x_1, \ldots, x_n) = f \quad \forall\, n\text{-ary } f \in \mathcal{F}$
- labeling
  - $L_f = \mathcal{F}^n$
  - $\ell_f(x_1, \ldots, x_n) = x_1 \cdots x_n$

Example

$\mathcal{R}$ $\quad \mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$

$$\begin{aligned} \mathsf{a_b}(\mathsf{b_a}(x)) &\to \mathsf{c_a}(x) \\ \mathsf{a_b}(\mathsf{b_b}(x)) &\to \mathsf{c_b}(x) \\ \mathsf{a_b}(\mathsf{b_c}(x)) &\to \mathsf{c_c}(x) \end{aligned} \qquad \mathcal{R}_{\mathsf{rlab}}$$

Problem

no model: $\forall\, \alpha \quad [\alpha](\mathsf{a}(\mathsf{b}(x))) = \mathsf{a} \neq \mathsf{c} = [\alpha](\mathsf{c}(x))$

Definitions (TRS $\mathcal{R}$ over signature $\mathcal{F}$)

- root-preserving rules
  $\mathcal{R}_{\mathsf{p}} = \{\, \ell \to r \in \mathcal{R} \mid \mathrm{root}(\ell) = \mathrm{root}(r) \,\}$
- root-altering rules
  $\mathcal{R}_{\mathsf{a}} = \{\, \ell \to r \in \mathcal{R} \mid \mathrm{root}(\ell) \neq \mathrm{root}(r) \,\}$
- flat contexts
  $\mathcal{FC} = \{\, g(x_1, \ldots, x_{i-1}, \Box, x_i, \ldots, x_n) \mid g \in \mathcal{F} \text{ has arity } n \text{ and } 1 \leqslant i \leqslant n \,\}$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}_{\mathsf{p}} \cup \{\, C[\ell] \to C[r] \mid \ell \to r \in \mathcal{R}_{\mathsf{a}} \text{ and } C \in \mathcal{FC} \,\}$

Lemma

- *$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})$ is terminating*
- *$\mathcal{A}_{\mathcal{F}}$ is model of $\mathcal{FC}(\mathcal{R})$*

Example

$\mathcal{R}$ $\quad$ $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{c}(x)$

$\mathcal{FC}(\mathcal{R})$

$$\mathsf{a}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{a}(\mathsf{c}(x)) \qquad \mathsf{a}(\Box)$$
$$\mathsf{b}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{b}(\mathsf{c}(x)) \qquad \mathsf{b}(\Box)$$
$$\mathsf{c}(\mathsf{a}(\mathsf{b}(x))) \to \mathsf{c}(\mathsf{c}(x)) \qquad \mathsf{c}(\Box)$$

$\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$

$$\mathsf{a}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{a}(x))) \to \mathsf{a}_\mathsf{c}(\mathsf{c}_\mathsf{a}(x))$$
$$\mathsf{a}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{b}(x))) \to \mathsf{a}_\mathsf{c}(\mathsf{c}_\mathsf{b}(x))$$
$$\mathsf{a}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{c}(x))) \to \mathsf{a}_\mathsf{c}(\mathsf{c}_\mathsf{c}(x))$$
$$\mathsf{b}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{a}(x))) \to \mathsf{b}_\mathsf{c}(\mathsf{c}_\mathsf{a}(x))$$
$$\mathsf{b}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{b}(x))) \to \mathsf{b}_\mathsf{c}(\mathsf{c}_\mathsf{b}(x))$$
$$\mathsf{b}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{c}(x))) \to \mathsf{b}_\mathsf{c}(\mathsf{c}_\mathsf{c}(x))$$
$$\mathsf{c}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{a}(x))) \to \mathsf{c}_\mathsf{c}(\mathsf{c}_\mathsf{a}(x))$$
$$\mathsf{c}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{b}(x))) \to \mathsf{c}_\mathsf{c}(\mathsf{c}_\mathsf{b}(x))$$
$$\mathsf{c}_\mathsf{a}(\mathsf{a}_\mathsf{b}(\mathsf{b}_\mathsf{c}(x))) \to \mathsf{c}_\mathsf{c}(\mathsf{c}_\mathsf{c}(x))$$

Corollary

$\mathcal{R}$ is terminating if and only if $\mathcal{FC}(\mathcal{R})_{rlab}$ is terminating

Example (1)

- $\mathcal{R}$ $\quad$ $\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$
  $$\mathsf{f}_{(\mathsf{a},\mathsf{b},\mathsf{a})}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_{(\mathsf{a},\mathsf{a},\mathsf{a})}(x,x,x)$$
  $$\mathsf{f}_{(\mathsf{a},\mathsf{b},\mathsf{b})}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_{(\mathsf{b},\mathsf{b},\mathsf{b})}(x,x,x)$$
  $$\mathsf{f}_{(\mathsf{a},\mathsf{b},\mathsf{f})}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_{(\mathsf{f},\mathsf{f},\mathsf{f})}(x,x,x)$$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ is terminating because it lacks dependency pairs

Example (2)

- $\mathcal{R}$ $\quad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$
- $\mathcal{FC}(\mathcal{R}) = \mathcal{R}$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ $\quad \mathsf{f}_{\mathsf{f}}(\mathsf{f}_{\mathsf{f}}(x)) \to \mathsf{f}_{\mathsf{g}}(\mathsf{g}_{\mathsf{f}}(\mathsf{f}_{\mathsf{f}}(x)))$
  $\mathsf{f}_{\mathsf{f}}(\mathsf{f}_{\mathsf{g}}(x)) \to \mathsf{f}_{\mathsf{g}}(\mathsf{g}_{\mathsf{f}}(\mathsf{f}_{\mathsf{g}}(x)))$
- $\mathcal{FC}(\mathcal{R})_{\mathsf{rlab}}$ is polynomially terminating

Outline

Definitions

- derivation length $\mathrm{dl}(t) = \max\{\, n \mid \exists\, u\colon t \to^n u \,\}$
- derivational complexity $\mathrm{dc}(n) = \max\{\, \mathrm{dl}(t) \mid |t| = n \,\}$

Example

rewrite rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x))$

term	derivation length	value
$\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))$	3	6
$\mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{c}))))))$	9	24
$\mathsf{a}^m(\mathsf{b}^n(\mathsf{c}))$	mn	$2^m n$

polynomial interpretation

$$\mathsf{a}_{\mathbb{N}}(x) = 2x \qquad \mathsf{b}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{c}_{\mathbb{N}} = 0$$

Theorem

	interpretation in $\mathbb{N}$	*bound on derivational complexity*
	polynomial	*double-exponential*
$a_1 x_1 + \cdots + a_n x_n + b$	*linear*	*exponential*
$x_1 + \cdots + x_n + b$	*strongly linear*	*linear*

Example

rewrite system

$$x + 0 \to x \qquad \mathsf{d}(0) \to 0 \qquad \mathsf{q}(0) \to 0$$

$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad \mathsf{d}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \qquad \mathsf{q}(\mathsf{s}(x)) \to \mathsf{q}(x) + \mathsf{s}(\mathsf{d}(x))$$

interpretations

$$0_{\mathbb{N}} = 2 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad +_{\mathbb{N}}(x, y) = x + 2y \quad \mathsf{d}_{\mathbb{N}}(x) = 3x \quad \mathsf{q}_{\mathbb{N}}(x) = x^3$$

Outline

Definition

triangular matrix interpretation $\mathcal{M}$ over $\mathbb{N}$

- carrier of $\mathcal{M}$ is $\mathbb{N}^d$ with $d > 0$
- interpretations (for every n-ary f)

$$f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n) = M_1\vec{x}_1 + \cdots + M_n\vec{x}_n + \vec{f}$$

  with

  - matrices $M_1, \ldots, M_n \in \mathbb{N}^{d \times d}$ with

$$(M_i)_{1,1} \geqslant 1 \qquad (M_i)_{j,k} = 0 \text{ for all } j > k \qquad (M_i)_{j,j} \leqslant 1 \text{ for all } j$$

    for all $1 \leqslant i \leqslant n$
  - vector $\vec{f} \in \mathbb{N}^d$

- $(x_1, \ldots, x_d)^{\mathsf{T}} > (y_1, \ldots, y_d)^{\mathsf{T}} \iff x_1 > y_1 \;\wedge\; \bigwedge_{i=2}^{d} x_i \geqslant y_i$

Theorem

if $\mathcal{R} \subseteq >_{\mathcal{M}}$ for triangular matrix interpretation $\mathcal{M}$ of dimension d then

$$\mathsf{dc}(n) = \mathcal{O}(n^d)$$

Example

rewrite rules

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x)) \qquad \mathsf{c}(\mathsf{a}(x)) \to \mathsf{b}(\mathsf{c}(x)) \qquad \mathsf{c}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{c}(x))$$

triangular matrix interpretation

$$\mathsf{a}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathsf{c}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathsf{b}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Outline

Matrix Interpretations for Proving Termination of Term Rewriting
Jörg Endrullis, Johannes Waldmann and Hans Zantema
JAR 40(2,3), pp. 195 – 220, 2008

Mechanizing and Improving Dependency Pairs
Jürgen Giesl, René Thiemann, Peter Schneider-Kamp and Stephan Falke
JAR 37(3), pp. 155 – 203, 2006

Tyrolean Termination Tool: Techniques and Features
Nao Hirokawa and Aart Middeldorp
I&C 205(4), pp. 474 – 511, 2007

Termination of Term Rewriting by Semantic Labelling
Hans Zantema
FI 24(1,2), pp. 89 – 105, 1995

Root-Labeling
Christian Sternagel and Aart Middeldorp
Proc. 19th RTA, LNCS 5117, pp. 336 – 350, 2008

Complexity Analysis of Term Rewriting Based on Matrix and Context Dependent Interpretations
Georg Moser, Andreas Schnabl and Johannes Waldmann
Proc. 28th FSTTCS, LIPI 2, pp. 304 – 315, 2008

Termination Tools

- AProVE
- CiME
- Jambox
- Matchbox
- MuTerm
- TTT2
- VMTL
- ...

Complexity Tools

- CaT
- TCT

Definition

- $\text{tcap}_{\mathcal{R}}(t) = f(\text{tcap}_{\mathcal{R}}(t_1), \ldots, \text{tcap}_{\mathcal{R}}(t_n))$

  if $t = f(t_1, \ldots, t_n)$ and $f(\text{tcap}_{\mathcal{R}}(t_1), \ldots, \text{tcap}_{\mathcal{R}}(t_n))$ does not unify with any left-hand side of $\mathcal{R}$
- $\text{tcap}_{\mathcal{R}}(t)$ is fresh variable

  otherwise

Definition (Modern Approximation)

$$s \to t \xrightarrow{\mathsf{m}} u \to v \quad \iff \quad \begin{cases} \text{tcap}_{\mathcal{R}}(t) \text{ and } u \text{ are unifiable} \\ t \text{ and } \text{tcap}_{\mathcal{R}^{-1}}(u) \text{ are unifiable} \end{cases}$$