Introduction to Term Rewriting

lecture 1

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Outline

Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

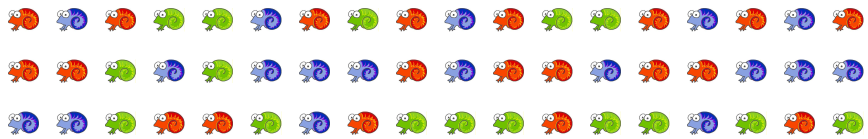
Wednesday

confluence, modularity, strategies

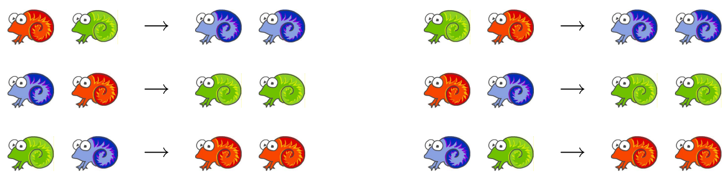
Thursday

exam, advanced topics

Outline



A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different colors meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?



A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

TAGCTAGCTAGCT

in every fertilized egg into the cola gene

CTGACTGACT

Techniques exist to perform the following DNA substitutions

TCAT ↔ T GAG ↔ AG CTC ↔ TC AGTA ↔ A TAT ↔ CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

CTGCTACTGACT

What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) s (unary) $+$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(x)+y$
rewrite rules	$0+y \to y$ $\mathsf{s}(x)+y \to \mathsf{s}(x+y)$
rewriting	$\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0+\mathsf{s}(\mathsf{s}(0))) \qquad x \mapsto 0 \quad y \mapsto \mathsf{s}(\mathsf{s}(0))$ $\to \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

Example (Addition on Natural Numbers in Decimal Notation)

signature: $0 \;\; 1 \;\; \cdots \;\; 9$ (constants) $\;+\;\; :\;$ (binary, infix)

terms: $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules:

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting: $(2:3)+(7:7) \to^{*} (1:0):0$ normal form

Example (Binary Trees)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)
leaf sum (unary) node (binary)

terms $\mathsf{leaf}((1:0):0) \quad \mathsf{node}(\mathsf{leaf}(1), \mathsf{leaf}(2)) \quad \mathsf{leaf}(\mathsf{node}(1, \mathsf{leaf}(2)))$

rewrite rules $\cdots$

$$\mathsf{sum}(\mathsf{leaf}(x)) \to x$$
$$\mathsf{sum}(\mathsf{node}(x,y)) \to \mathsf{sum}(x) + \mathsf{sum}(y)$$

rewriting

$$\begin{aligned} &\mathsf{sum}(\mathsf{node}(\mathsf{leaf}(2:3), \mathsf{leaf}(7:7))) \\ &\quad \to \mathsf{sum}(\mathsf{leaf}(2:3)) + \mathsf{sum}(\mathsf{leaf}(7:7)) \\ &\quad \to^* (2:3) + (7:7) \\ &\quad \to^* (1:0):0 \end{aligned}$$

Example (Group Theory)

signature e (constant) $\;^-$ (unary, postfix) $\;\cdot$ (binary, infix)

equations $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$

rewrite rules $\mathcal{R}$

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same $\mathcal{R}$-normal form

② $\mathcal{R}$ admits no infinite computations

① & ② $\Longrightarrow$ $\mathcal{E}$ has decidable validity problem

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K}\cdot\mathsf{I})\cdot\mathsf{I})\cdot\mathsf{S}$ $(x\cdot z)\cdot(y\cdot z)$

rewrite rules

$$\mathsf{I}\cdot x \to x$$
$$(\mathsf{K}\cdot x)\cdot y \to x$$
$$((\mathsf{S}\cdot x)\cdot y)\cdot z \to (x\cdot z)\cdot(y\cdot z)$$

rewriting $((\mathsf{S}\cdot\mathsf{K})\cdot\mathsf{K})\cdot x \;\to\; (\mathsf{K}\cdot x)\cdot(\mathsf{K}\cdot x) \;\to\; x$

inventor Moses Schönfinkel (1924)

Example (Lambda Calculus)

signature λ (binds variables) $\cdot$ (application, binary, infix)

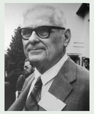
terms $M ::= x \mid (\lambda x.\, M) \mid (M\cdot M)$

α conversion $\lambda x.\, x\cdot y =_\alpha \lambda z.\, z\cdot y$

β reduction $(\lambda x.\, M)\cdot N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting $(\lambda x.\, x\cdot x)\cdot(\lambda x.\, x\cdot x) \;\to\; (\lambda x.\, x\cdot x)\cdot(\lambda x.\, x\cdot x)$



inventor Alonzo Church (1936)

Outline

- Overview
- Examples
- Terms
  - Contexts
  - Substitutions

Definition

- signature $\mathcal{F}$ function symbols with arities
- variables $\mathcal{V}$ $\mathcal{F} \cap \mathcal{V} = \varnothing$ infinitely many
- terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$ smallest set such that
  - $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$
  - if $f \in \mathcal{F}$ has arity $n \geqslant 0$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$
- ground terms $\mathcal{T}(\mathcal{F})$ smallest set such that
  - if $f \in \mathcal{F}$ has arity $n \geqslant 0$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F})$

Definitions (Operations on Terms)

- $\mathcal{V}\mathrm{ar}(\cdot)$ — $(2 : x) + ((1 : x) : y)$

$$\mathcal{V}\mathrm{ar}(t) = \begin{cases} \{t\} & \text{if } t \in \mathcal{V} \\ \bigcup_{i=1}^{n} \mathcal{V}\mathrm{ar}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- $\mathcal{F}\mathrm{un}(\cdot)$ — $(2 : x) + ((1 : x) : y)$

$$\mathcal{F}\mathrm{un}(t) = \begin{cases} \varnothing & \text{if } t \in \mathcal{V} \\ \{f\} \cup \bigcup_{i=1}^{n} \mathcal{F}\mathrm{un}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

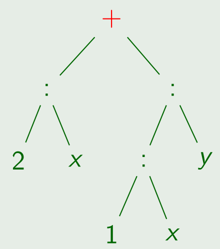
Definition (Operations on Terms)

- $\mathrm{root}(\cdot)$

$$\mathrm{root}(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Example

$(2 : x) + ((1 : x) : y)$



Definitions (Operations on Terms)

- $|\cdot|$ $\qquad |(2:x)+((1:x):y)| = 9$

$$|t| = \begin{cases} 1 & \text{if } t \in \mathcal{V} \\ 1 + \sum_{i=1}^{n} |t_i| & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

- $\|\cdot\|$ $\qquad \|(2:x)+((1:x):y)\| = 6$

$$\|t\| = \begin{cases} 0 & \text{if } t \in \mathcal{V} \\ 1 + \sum_{i=1}^{n} \|t_i\| & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

Definitions

- $s \trianglelefteq t$ $\quad$ s is subterm of t
  - $s = t$ or
  - $t = f(t_1,\dots,t_n)$ and $s \trianglelefteq t_i$ for some $1 \leqslant i \leqslant n$
- $s \triangleleft t$ $\quad$ s is proper subterm of t
  - $s \trianglelefteq t$ and $s \neq t$

Example

term $(2:x)+((1:x):y)$ has subterms

$2 \quad x \quad 2:x \quad 1 \quad 1:x \quad y \quad (1:x):y \quad (2:x)+((1:x):y)$

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Definitions

- context is term with one hole, i.e., element of $\mathcal{T}(\mathcal{F} \cup \{\square\}, \mathcal{V})$ that contains exactly one occurrence of $\square$
- $C[t]$ denotes result of replacing hole in context C by term t
- relation R on terms is closed under contexts if $\forall$ terms s, t

$$s \mathrel{R} t \quad \Longrightarrow \quad \forall \text{ contexts } C\colon \quad C[s] \mathrel{R} C[t]$$

Examples

- $\square \qquad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(\square)) \qquad \square + x$
- $\square[\mathsf{s}(0)] = \mathsf{s}(0) \qquad (\square + x)[0 + x] = (0 + x) + x$

Lemmata

- $s \trianglelefteq t \iff \exists \textit{ context } C\colon \quad t = C[s]$
- $s \triangleleft t \iff \exists \textit{ context } C \neq \square\colon \quad t = C[s]$

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Definitions

- substitution is mapping $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain

$$\mathcal{D}\mathrm{om}(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$$

  is finite
- empty substitution ε $\quad (\mathcal{D}\mathrm{om}(\varepsilon) = \varnothing)$
- application of substitution σ to term t

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \in \mathcal{V} \\ f(t_1\sigma, \dots, t_n\sigma) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

Example

$t = x + \mathsf{s}(y + z) \quad \sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\} \quad t\sigma = \mathsf{s}(y) + \mathsf{s}((x + \mathsf{s}(0)) + z)$

Definition

relation R on terms is closed under substitutions if $\forall$ terms s, t

$$s \; R \; t \quad \Longrightarrow \quad \forall \text{ substitutions } \sigma : \quad s\sigma \; R \; t\sigma$$

Definition (Matching Problem)

instance: terms s, t
question: $\exists$ subsitution σ: $s\sigma = t$? (s matches t)

Remarks

- term t can be rewritten if left-hand side of rewrite rule matches subterm of t
- matching problem is decidable (in linear time)

Definition (Matching Problem)

instance: terms s, t
question: $\exists$ subsitution σ: $s\sigma = t$?

Matching Algorithm

1. start with $\{s \mapsto t\}$
2. repeatedly apply following tranformation rules

$$\{f(s_1,\ldots,s_n) \mapsto f(t_1,\ldots,t_n)\} \cup S \;\Rightarrow\; \{s_1 \mapsto t_1,\ldots,s_n \mapsto t_n\} \cup S$$

$$\{f(s_1,\ldots,s_n) \mapsto g(t_1,\ldots,t_n)\} \cup S \;\Rightarrow\; \bot \;\text{ if } f \neq g$$

$$\{f(s_1,\ldots,s_n) \mapsto x\} \cup S \;\Rightarrow\; \bot$$

$$\{x \mapsto t\} \cup S \;\Rightarrow\; \bot \;\text{ if } S \text{ contains } x \mapsto t' \text{ with } t \neq t'$$

Examples

- $x + \mathsf{s}(y+z)$ matches $\mathsf{s}(y) + \mathsf{s}((x+\mathsf{s}(0))+z)$:

$$\begin{aligned} & \{x + \mathsf{s}(y+z) \mapsto \mathsf{s}(y) + \mathsf{s}((x+\mathsf{s}(0))+z)\} \\ \Longrightarrow\quad & \{x \mapsto \mathsf{s}(y),\ \mathsf{s}(y+z) \mapsto \mathsf{s}((x+\mathsf{s}(0))+z)\} \\ \Longrightarrow\quad & \{x \mapsto \mathsf{s}(y),\ y+z \mapsto (x+\mathsf{s}(0))+z\} \\ \Longrightarrow\quad & \{x \mapsto \mathsf{s}(y),\ y \mapsto x+\mathsf{s}(0),\ z \mapsto z\} \end{aligned}$$

- $x^- \cdot (x \cdot y)$ does not match $(\mathsf{e} \cdot x)^- \cdot ((\mathsf{e}\cdot\mathsf{e}) \cdot x)$:

$$\begin{aligned} & \{x^- \cdot (x \cdot y) \mapsto (\mathsf{e}\cdot x)^- \cdot ((\mathsf{e}\cdot\mathsf{e})\cdot x)\} \\ \Longrightarrow\quad & \{x^- \mapsto (\mathsf{e}\cdot x)^-,\ x\cdot y \mapsto (\mathsf{e}\cdot\mathsf{e})\cdot x\} \\ \Longrightarrow\quad & \{x \mapsto \mathsf{e}\cdot x,\ x\cdot y \mapsto (\mathsf{e}\cdot\mathsf{e})\cdot x\} \\ \Longrightarrow\quad & \{x \mapsto \mathsf{e}\cdot x,\ x \mapsto \mathsf{e}\cdot\mathsf{e},\ y \mapsto x\} \\ \Longrightarrow\quad & \bot \end{aligned}$$