Introduction to Term Rewriting

lecture 2

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Department of Computer Science
VU Amsterdam

Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Abstract Rewrite Systems
- Newman's Lemma
- Multiset Orders
- Further Reading

Motivation

concrete rewrite formalisms

- string rewriting

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abstract rewriting

- no structure on objects that are rewritten

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concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
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- interaction nets
- $\cdots$

abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Outline

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  - Definitions
  - Properties
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- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$

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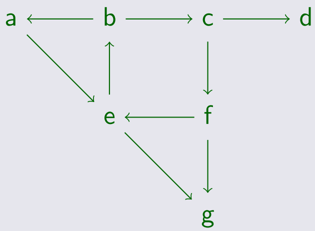

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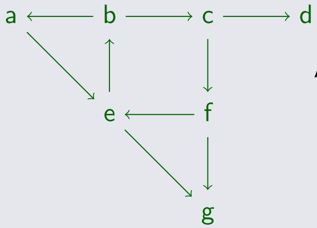

ARS $\mathcal{A} = \langle A, \to \rangle$

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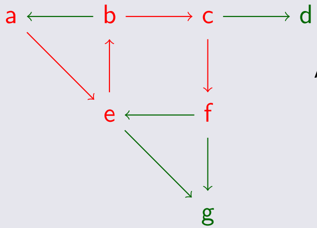

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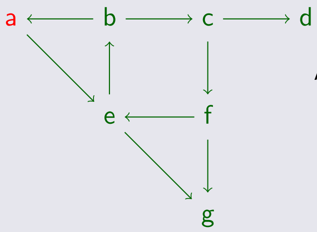

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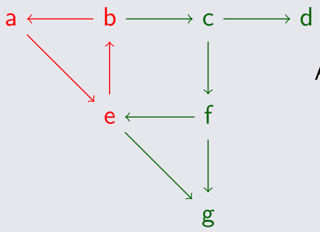

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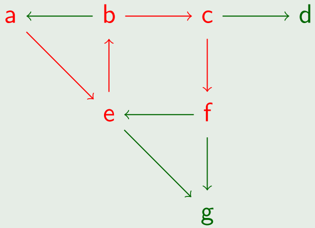

- $a \to^* f$

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- if $x \to^* z \,{}^*\!\!\leftarrow y$ then z is common reduct of x and y

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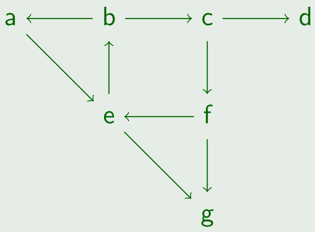

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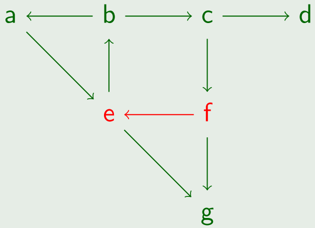

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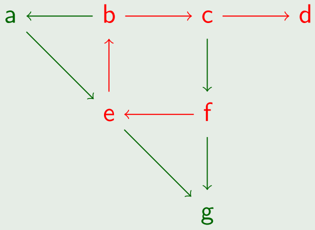

- $a \to^* f$
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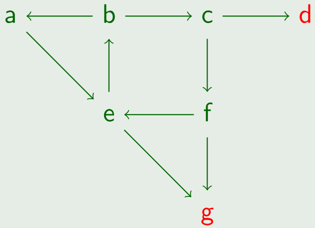

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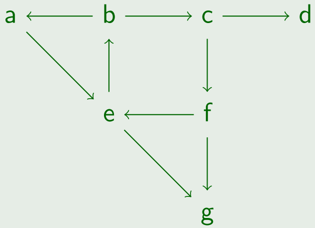

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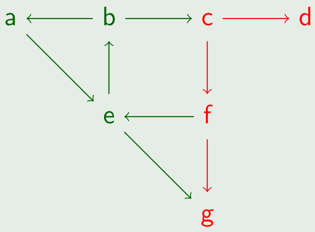

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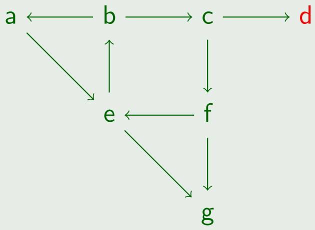

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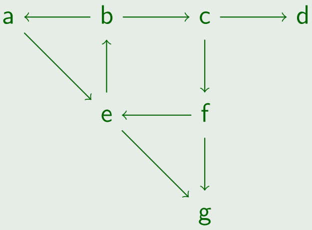

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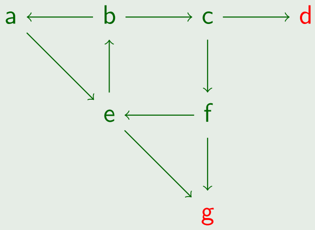

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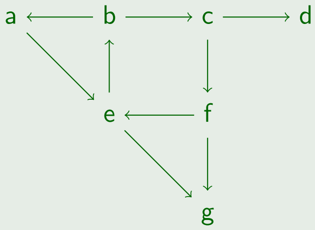

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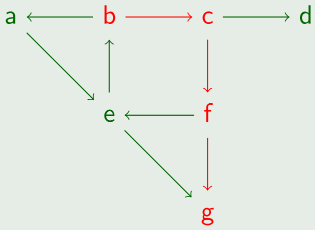

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  - Definitions
  - Properties
  - Relationships
- Newman's Lemma
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  - ${}^!\!\leftarrow \cdot \to^! \ \subseteq\ =$

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    - $\forall a, b, c$

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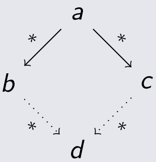

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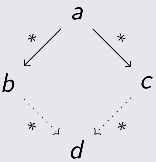

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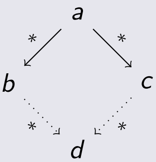

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2. $\mathrm{SN} \not\Longleftarrow \mathrm{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
3. $\mathrm{CR} \iff \leftrightarrow^{*} \subseteq \downarrow \iff \leftrightarrow^{*} = \downarrow$
4. $\mathrm{CR} \Longrightarrow \mathrm{UN}$

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4. CR $\Longrightarrow$ UN
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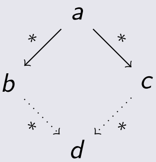

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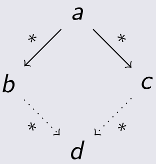

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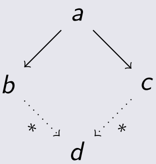

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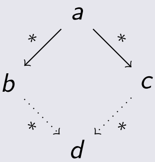

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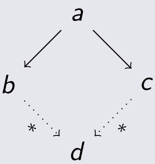

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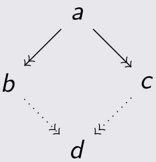

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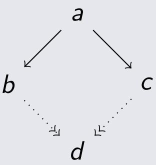

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in diagrams: $\twoheadrightarrow$ for $\rightarrow^*$

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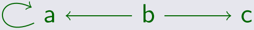

6. WN & UN $\Longrightarrow$ CR
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Lemmata

1. $\mathsf{SN} \Longrightarrow \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
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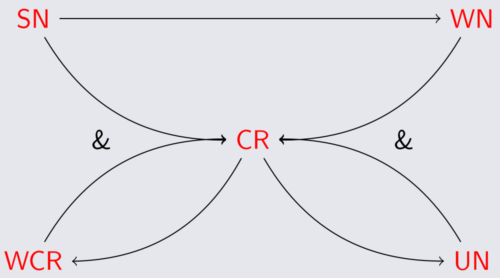
Summary



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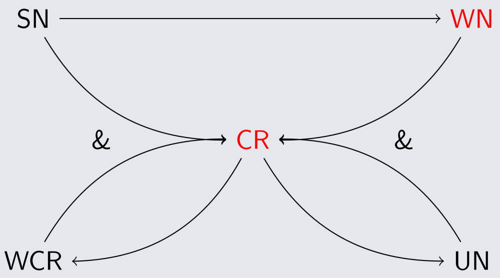

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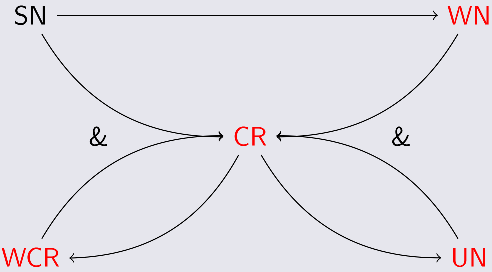

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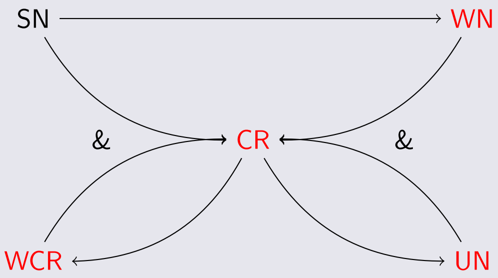

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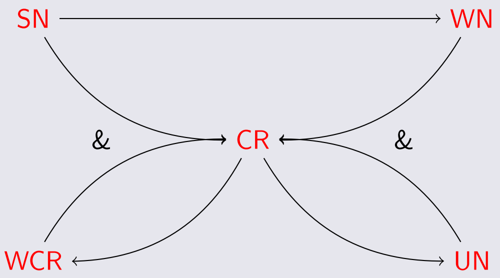

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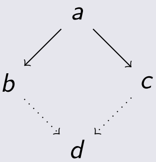

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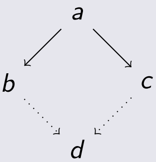

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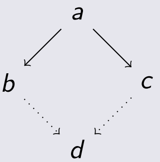

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Lemma

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow \ \subseteq \ \rightarrow_\diamond \ \subseteq \ \rightarrow^*$ for some relation $\rightarrow_\diamond$ on A with diamond property

Outline

- Abstract Rewrite Systems
- Newman's Lemma
- Multiset Orders
- Further Reading

Well-Founded Induction

given

- property P of ARSs that satisfies $\mathsf{P}(\mathcal{A}) \iff \forall a\colon \mathsf{P}(a)$

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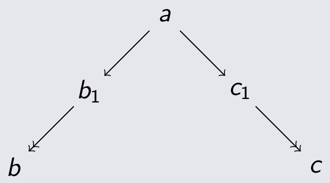
$$\Big(\forall a\colon \big(\forall b\colon a \to b \implies \mathsf{P}(b)\big) \implies \mathsf{P}(a)\Big) \implies \forall a\colon \mathsf{P}(a)$$

Newman's Lemma

$\mathsf{SN}(\mathcal{A})$ & $\mathsf{WCR}(\mathcal{A}) \implies \mathsf{CR}(\mathcal{A})$

First Proof



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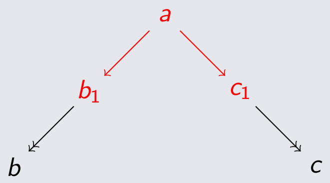

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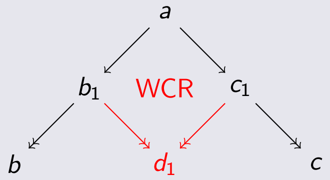

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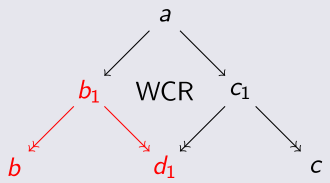

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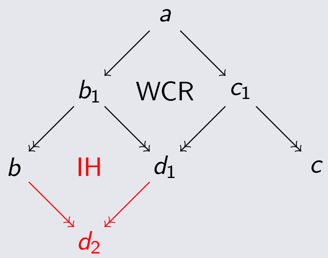

induction hypothesis $\mathrm{CR}(b_1)$

$\forall a'$: if $a \to a'$ then $\mathrm{CR}(a')$

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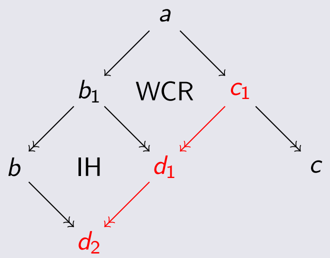

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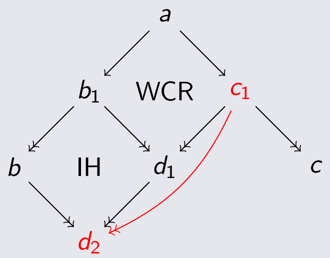

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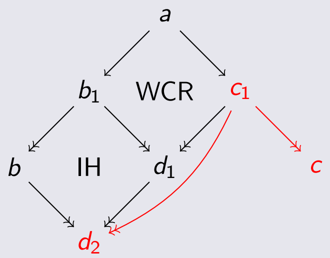

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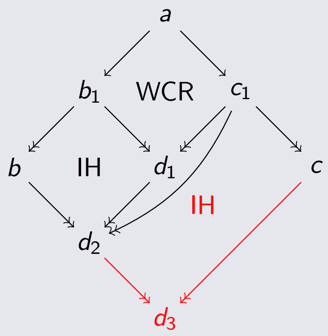

induction hypothesis $\quad$ $\mathsf{CR}(c_1)$

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$\{a, a, a, b, d, d, d\} \in \mathcal{M}(\{a, b, c, d\})$:

$$a \mapsto 3 \qquad b \mapsto 1 \qquad c \mapsto 0 \qquad d \mapsto 3$$

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Theorem

- *multiset extension of proper order is proper order*

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$$\{2,3\} >_{\mathsf{mul}} \{0,1,3\} >_{\mathsf{mul}} \{0,1,1,2,2,2\} >_{\mathsf{mul}} \{0,1,1,0,1,1,2,2\}$$
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$$>_{\mathsf{mul}} \{1,1,1,1\} >_{\mathsf{mul}} \{0,0,0,0,1,1,1\} >_{\mathsf{mul}} \cdots$$

Theorem

- *multiset extension of proper order is proper order*
- *multiset extension of well-founded order is well-founded*

Newman's Lemma

$\mathsf{SN}(\mathcal{A})$ & $\mathsf{WCR}(\mathcal{A}) \quad \Longrightarrow \quad \mathsf{CR}(\mathcal{A})$

Second Proof

- given $b \; {}^*\!\!\leftarrow a \rightarrow^* c$

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Second Proof

- given $b \ {}^*\!\!\leftarrow a \rightarrow^* c$
- construct sequence of conversions $(C_i)_{i \geqslant 0}$ between b and c

Newman's Lemma

SN($\mathcal{A}$) & WCR($\mathcal{A}$) $\implies$ CR($\mathcal{A}$)

Second Proof

- given $b \;{}^*\!\!\leftarrow a \rightarrow^* c$
- construct sequence of conversions $(C_i)_{i \geqslant 0}$ between b and c
  - C_0 is initial conversion $b \;{}^*\!\!\leftarrow a \rightarrow^* c$

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$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(\mathcal{A})$

Second Proof

- given $b \mathrel{{}^*{\leftarrow}} a \to^* c$
- construct sequence of conversions $(C_i)_{i \geqslant 0}$ between b and c
  - C_0 is initial conversion $b \mathrel{{}^*{\leftarrow}} a \to^* c$
  - C_{i+1} is obtained from C_i by replacing peak $e \leftarrow d \to f$ in C_i by valley $e \to^* \cdot \mathrel{{}^*{\leftarrow}} f$

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- $|C_i|$ is multiset of elements appearing in C_i

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- $|C_i|$ is multiset of elements appearing in C_i
- $|C_i| \; (\rightarrow^+)_{\mathsf{mul}} \; |C_{i+1}|$

Newman's Lemma

$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(\mathcal{A})$

Second Proof

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- $|C_i|$ is multiset of elements appearing in C_i
- $|C_i|\ (\to^+)_{\mathrm{mul}}\ |C_{i+1}|$
- $(\to^+)_{\mathrm{mul}}$ is well-founded

Newman's Lemma

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- $|C_i|$ is multiset of elements appearing in C_i
- $|C_i| \; (\to^+)_{\mathrm{mul}} \; |C_{i+1}|$
- $(\to^+)_{\mathrm{mul}}$ is well-founded
- hence $\exists n$ such that C_n has no peaks

Newman's Lemma

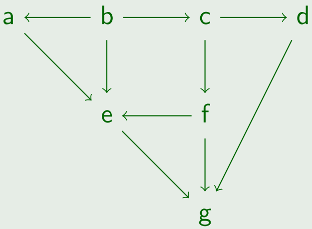
$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(\mathcal{A})$

Second Proof

- given $b \, {}^*\!\!\leftarrow a \rightarrow^* c$
- construct sequence of conversions $(C_i)_{i \geqslant 0}$ between b and c
  - C_0 is initial conversion $b \, {}^*\!\!\leftarrow a \rightarrow^* c$
  - C_{i+1} is obtained from C_i by replacing peak $e \leftarrow d \rightarrow f$ in C_i by valley $e \rightarrow^* \cdot \, {}^*\!\!\leftarrow f$
- $|C_i|$ is multiset of elements appearing in C_i
- $|C_i| \; (\rightarrow^+)_{\mathrm{mul}} \; |C_{i+1}|$
- $(\rightarrow^+)_{\mathrm{mul}}$ is well-founded
- hence $\exists n$ such that C_n has no peaks $\implies C_n \colon b \rightarrow^* \cdot \, {}^*\!\!\leftarrow c$

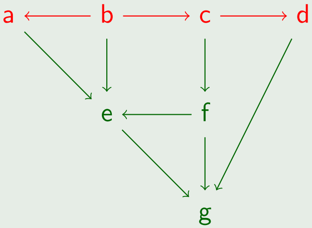
Example

- ARS



Example

- ARS

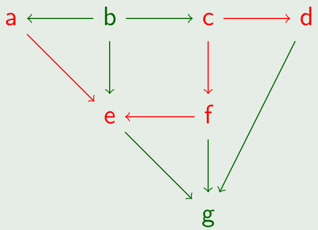


- conversion multiset

$a \leftarrow b \rightarrow c \rightarrow d$ $\{a, b, c, d\}$

Example

- ARS

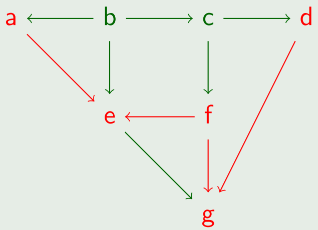


- conversion multiset

conversion	multiset
$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$
$a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$	$\{a, e, f, c, d\}$

Example

- ARS

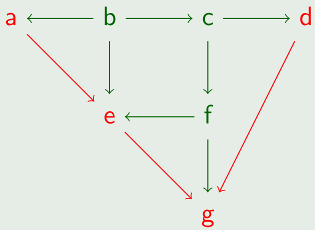


- conversion multiset

conversion	multiset
$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$
$a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$	$\{a, e, f, c, d\}$
$a \rightarrow e \leftarrow f \rightarrow g \leftarrow d$	$\{a, e, f, g, d\}$

Example

- ARS



- conversion — multiset

conversion	multiset	
$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$	
$a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$	$\{a, e, f, c, d\}$	
$a \rightarrow e \leftarrow f \rightarrow g \leftarrow d$	$\{a, e, f, g, d\}$	
$a \rightarrow e \rightarrow g \leftarrow d$	$\{a, e, g, d\}$	rewrite proof

Outline

- Abstract Rewrite Systems
- Newman's Lemma
- Multiset Orders
- Further Reading

Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems
Gérard Huet
JACM 27(4), pp. 797 – 821, 1980

Proving Termination with Multiset Orderings
Nachum Dershowitz and Zohar Manna
CACM 22(8), pp. 465 – 476, 1979

Confluence by Decreasing Diagrams
Vincent van Oostrom
TCS 126(2), pp. 259 – 280, 1994