ISR 2010

Introduction to Term Rewriting

lecture 2

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Abstract Rewrite Systems
- Newman's Lemma
- Multiset Orders
- Further Reading

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- $\dots$

abstract rewriting

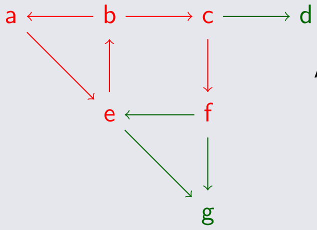
- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Outline

- Abstract Rewrite Systems
  - Definitions
  - Properties
  - Relationships
- Newman's Lemma
- Multiset Orders
- Further Reading

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$



ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to = \left\{ \begin{matrix} (a,e), (b,a), (b,c), (c,d), (c,f) \\ (e,b), (e,g), (f,e), (f,g) \end{matrix} \right\}$

- rewrite sequence
  - finite $a \to e \to b \to c \to f$
  - empty a
  - infinite $a \to e \to b \to a \to e \to b \to \cdots$

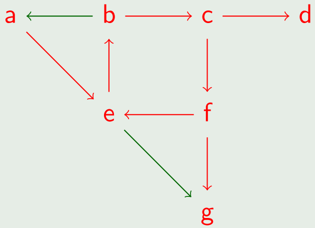
Definition (Derived Relations of $\rightarrow$)

- $\leftarrow$ inverse of $\rightarrow$
- $\rightarrow^*$ transitive and reflexive closure of $\rightarrow$
- ${}^*\!\leftarrow$ inverse of $\rightarrow^*$ (transitive and reflexive closure of $\leftarrow$)
- $\downarrow$ joinability $\downarrow \, = \, \rightarrow^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\rightarrow$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\rightarrow$)
- $\rightarrow^+$ transitive closure of $\rightarrow$
- $\rightarrow^=$ reflexive closure of $\rightarrow$
- $\uparrow$ meetability $\uparrow \, = \, {}^*\!\leftarrow \cdot \rightarrow^*$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \; {}^*\!\leftarrow y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are convertible

Example

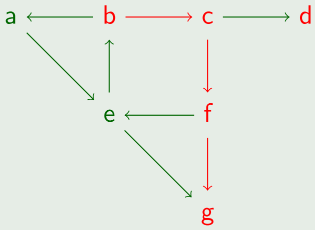


- a $\to^*$ f
- e $\downarrow$ f f $\downarrow$ d not g $\downarrow$ d
- g $\leftrightarrow^*$ d

Definition (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- $\mathrm{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$
- $x \to^! y$ if $x \to^* y$ for normal form y (x has normal form y)

Example



ARS $\mathcal{A} = \langle A, \to \rangle$

- d is normal form
- $\mathrm{NF}(\mathcal{A}) = \{\mathrm{d}, \mathrm{g}\}$
- $\mathrm{b} \to^! \mathrm{g}$

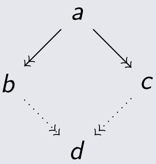
Outline

Definitions

- SN strong normalization termination
  - no infinite rewrite sequences
- WN weak normalization
  - every element has at least one normal form
  - $\forall a\ \exists b \quad a \rightarrow^! b$
- UN unique normal forms
  - no element has more than one normal form
  - $\forall a, b, c \quad$ if $a \rightarrow^! b$ and $a \rightarrow^! c$ then $b = c$
  - ${}^!\!\leftarrow \cdot \rightarrow^! \ \subseteq\ =$

Definitions

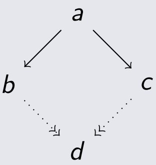
- CR   confluence   Church-Rosser property
  - $\uparrow \subseteq \downarrow$
  - $\forall a, b, c$



$\exists d$

Wikimedia

- WCR   local confluence   weak Church-Rosser property
  - $\leftarrow \cdot \rightarrow \subseteq \downarrow$
  - $\forall a, b, c$



$\exists d$

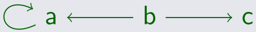
in diagrams: $\twoheadrightarrow$ for $\rightarrow^*$

Outline

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN

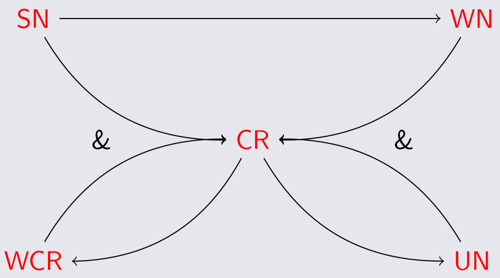
3. CR $\Longleftrightarrow$ $\leftrightarrow^* \subseteq \downarrow$ $\Longleftrightarrow$ $\leftrightarrow^* = \downarrow$
4. CR $\Longrightarrow$ UN
5. CR $\not\Longleftarrow$ UN



6. WN & UN $\Longrightarrow$ CR
7. CR $\Longrightarrow$ WCR
8. CR $\not\Longleftarrow$ WCR

9. SN & WCR $\Longrightarrow$ CR — Newman's Lemma

Summary

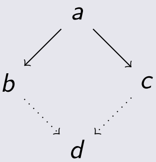


Definitions

- semi-completeness
  - CR & WN
  - every element has unique normal form
- completeness
  - CR & SN

Definition

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \ \subseteq \ \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$



$\exists d$

Lemma

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow \ \subseteq \rightarrow_\diamond \subseteq \ \rightarrow^*$ for some relation $\rightarrow_\diamond$ on A with diamond property

Outline

Well-Founded Induction

given

- property P of ARSs that satisfies $\quad \mathrm{P}(\mathcal{A}) \quad \Longleftrightarrow \quad \forall a\colon \mathrm{P}(a)$
- strongly normalizing ARS $\mathcal{A} = \langle A, \rightarrow \rangle$

to conclude

- $\mathrm{P}(\mathcal{A})$

it is sufficient to prove

- if $\underbrace{\mathrm{P}(b) \text{ for every } b \text{ with } a \rightarrow b}_{\text{induction hypothesis}}$ then $\mathrm{P}(a)$

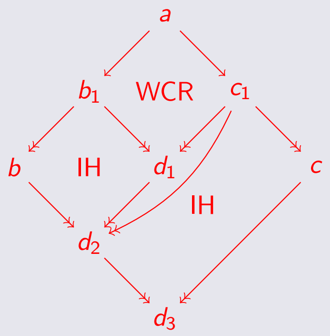
for arbitrary element a

$$\left(\forall a\colon \left(\forall b\colon a \rightarrow b \implies \mathrm{P}(b)\right) \implies \mathrm{P}(a)\right) \quad \implies \quad \forall a\colon \mathrm{P}(a)$$

Newman's Lemma

$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(\mathcal{A})$

First Proof



induction hypothesis $\mathrm{CR}(b_1)$ & $\mathrm{CR}(c_1)$

$\forall a'$: if $a \to a'$ then $\mathrm{CR}(a')$

Outline

Definitions (Multiset)

- finite multiset M over A is function from A to $\mathbb{N}$ such that $M(a) = 0$ for all but finitely many $a \in A$
- $M(a)$ is multiplicity of a
- set of all finite multisets over A is denoted by $\mathcal{M}(A)$

Example

$\{a, a, a, b, d, d, d\} \in \mathcal{M}(\{a, b, c, d\})$:

$$a \mapsto 3 \qquad b \mapsto 1 \qquad c \mapsto 0 \qquad d \mapsto 3$$

Definitions (Operations on Multisets)

- sum $\forall a\colon (M_1 \uplus M_2)(a) = M_1(a) + M_2(a)$
- difference $\forall a\colon (M_1 - M_2)(a) = \max\{M_1(a) - M_2(a), 0\}$
- $\ldots$

Example

- $\{a, a, a, b, d, d, d\} \uplus \{a, c, c\} = \{a, a, a, a, b, c, c, d, d, d\}$
- $\{a, a, a, b, d, d, d\} - \{a, c, c\} = \{a, a, b, d, d, d\}$

Definition

multiset extension of proper order $>$ on A is relation $>_{\mathsf{mul}}$ defined on $\mathcal{M}(A)$ as follows: $M_1 >_{\mathsf{mul}} M_2$ if $\exists X, Y \in \mathcal{M}(A)$ such that

- $M_2 = (M_1 - X) \uplus Y$
- $\varnothing \neq X \subseteq M_1$
- $\forall y \in Y \; \exists x \in X : x > y$

Example

$$\{2,3\} >_{\mathsf{mul}} \{0,1,3\} >_{\mathsf{mul}} \{0,1,1,2,2,2\} >_{\mathsf{mul}} \{0,1,1,0,1,1,2,2\}$$
$$>_{\mathsf{mul}} \{0,1,0,1,1,2,2\} >_{\mathsf{mul}} \{0,1,0,1,0,0,2\} >_{\mathsf{mul}} \{1,1,1,1,1,1\}$$
$$>_{\mathsf{mul}} \{1,1,1,1\} >_{\mathsf{mul}} \{0,0,0,0,1,1,1\} >_{\mathsf{mul}} \cdots$$

Theorem

- *multiset extension of proper order is proper order*
- *multiset extension of well-founded order is well-founded*

Newman's Lemma

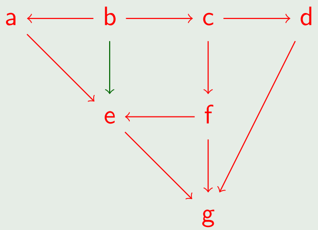
$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(\mathcal{A})$

Second Proof

- given $b\ {}^*\!\!\leftarrow a \rightarrow^* c$
- construct sequence of conversions $(C_i)_{i\geqslant 0}$ between b and c
  - C_0 is initial conversion $b\ {}^*\!\!\leftarrow a \rightarrow^* c$
  - C_{i+1} is obtained from C_i by replacing peak $e \leftarrow d \rightarrow f$ in C_i by valley $e \rightarrow^* \cdot\ {}^*\!\!\leftarrow f$
- $|C_i|$ is multiset of elements appearing in C_i
- $|C_i|\ (\rightarrow^+)_{\mathsf{mul}}\ |C_{i+1}|$
- $(\rightarrow^+)_{\mathsf{mul}}$ is well-founded
- hence $\exists n$ such that C_n has no peaks $\implies C_n\colon b \rightarrow^* \cdot\ {}^*\!\!\leftarrow c$

Example

- ARS



- conversion — multiset

conversion	multiset	
$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$	
$a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$	$\{a, e, f, c, d\}$	
$a \rightarrow e \leftarrow f \rightarrow g \leftarrow d$	$\{a, e, f, g, d\}$	
$a \rightarrow e \rightarrow g \leftarrow d$	$\{a, e, g, d\}$	rewrite proof

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- Newman's Lemma
- Multiset Orders
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Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems

Gérard Huet

JACM 27(4), pp. 797 – 821, 1980

Proving Termination with Multiset Orderings

Nachum Dershowitz and Zohar Manna

CACM 22(8), pp. 465 – 476, 1979

Confluence by Decreasing Diagrams

Vincent van Oostrom

TCS 126(2), pp. 259 – 280, 1994