Introduction to Term Rewriting

lecture 2

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- . . .

abstract rewriting

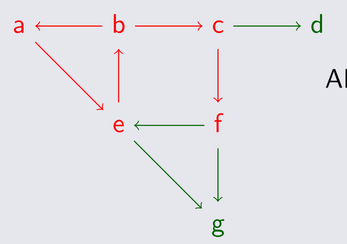
- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Outline

- Abstract Rewrite Systems
  - Definitions
  - Properties
  - Relationships
- Newman's Lemma
- Multiset Orders
- Further Reading

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$



ARS $\mathcal{A} = \langle A, \rightarrow \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\rightarrow = \begin{Bmatrix} (a,e),(b,a),(b,c),(c,d),(c,f) \\ (e,b),(e,g),(f,e),(f,g) \end{Bmatrix}$

- rewrite sequence
  - finite $\quad a \rightarrow e \rightarrow b \rightarrow c \rightarrow f$
  - empty $\quad a$
  - infinite $\quad a \rightarrow e \rightarrow b \rightarrow a \rightarrow e \rightarrow b \rightarrow \cdots$

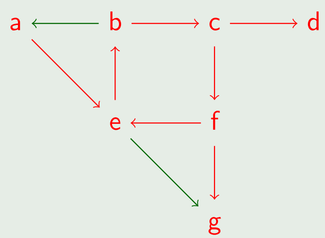
Definition (Derived Relations of $\to$)

- $\leftarrow$ inverse of $\to$
- $\to^*$ transitive and reflexive closure of $\to$
- ${}^*\!\leftarrow$ inverse of $\to^*$ (transitive and reflexive closure of $\leftarrow$)
- $\downarrow$ joinability $\downarrow\; = \;\to^* \cdot {}^*\!\leftarrow$
- $\leftrightarrow$ symmetric closure of $\to$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\to$)
- $\to^+$ transitive closure of $\to$
- $\to^=$ reflexive closure of $\to$
- $\uparrow$ meetability $\uparrow\; = \;{}^*\!\leftarrow \cdot \to^*$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \;{}^*\!\leftarrow y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are convertible

Example

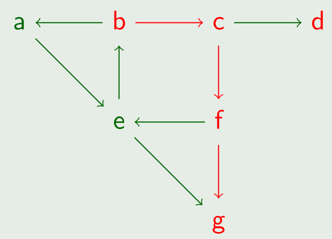


- a $\to^*$ f
- e $\downarrow$ f f $\downarrow$ d not g $\downarrow$ d
- g $\leftrightarrow^*$ d

Definition (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- $\mathsf{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$
- $x \to^! y$ if $x \to^* y$ for normal form y (x has normal form y)

Example



ARS $\mathcal{A} = \langle A, \to \rangle$

- d is normal form
- $\mathsf{NF}(\mathcal{A}) = \{\mathsf{d}, \mathsf{g}\}$
- $\mathsf{b} \to^! \mathsf{g}$

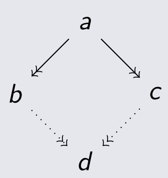
Outline

Definitions

- SN strong normalization termination
  - no infinite rewrite sequences
- WN weak normalization
  - every element has at least one normal form
  - $\forall a \, \exists b \quad a \to^! b$
- UN unique normal forms
  - no element has more than one normal form
  - $\forall a, b, c$ if $a \to^! b$ and $a \to^! c$ then $b = c$
  - ${}^!\!\leftarrow \cdot \to^! \ \subseteq \ =$

Definitions

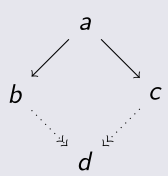
- CR confluence Church-Rosser property
  - $\uparrow \ \subseteq \ \downarrow$
  - $\forall a, b, c$



$\exists d$

Wikimedia

- WCR local confluence weak Church-Rosser property
  - $\leftarrow \cdot \to \ \subseteq \ \downarrow$
  - $\forall a, b, c$



in diagrams: $\twoheadrightarrow$ for $\to^*$

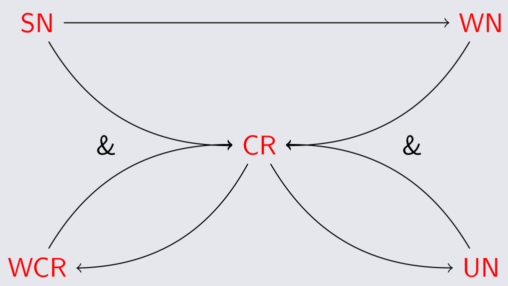
$\exists d$

Outline

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN $\circlearrowleft a \longrightarrow b$
3. CR $\iff$ $\leftrightarrow^* \subseteq \downarrow$ $\iff$ $\leftrightarrow^* = \downarrow$
4. CR $\Longrightarrow$ UN
5. CR $\not\Longleftarrow$ UN $\circlearrowleft a \longleftarrow b \longrightarrow c$
6. WN & UN $\Longrightarrow$ CR
7. CR $\Longrightarrow$ WCR
8. CR $\not\Longleftarrow$ WCR $a \longleftarrow b \rightleftarrows c \longrightarrow d$
9. SN & WCR $\Longrightarrow$ CR Newman's Lemma

Summary

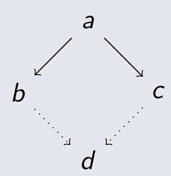


Definitions

- semi-completeness
  - CR & WN
  - every element has unique normal form
- completeness
  - CR & SN

Definition

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \ \subseteq \ \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$



$\exists d$

Lemma

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ *is confluent if* $\rightarrow \ \subseteq \rightarrow_{\diamond} \subseteq \ \rightarrow^{*}$ *for some relation* $\rightarrow_{\diamond}$ *on* A *with diamond property*

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- Abstract Rewrite Systems
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Well-Founded Induction

given

- property P of ARSs that satisfies $\mathsf{P}(\mathcal{A}) \iff \forall a\colon \mathsf{P}(a)$
- strongly normalizing ARS $\mathcal{A} = \langle A, \to \rangle$

to conclude

- $\mathsf{P}(\mathcal{A})$

it is sufficient to prove

- if $\underbrace{\mathsf{P}(b) \text{ for every } b \text{ with } a \to b}_{\text{induction hypothesis}}$ then $\mathsf{P}(a)$

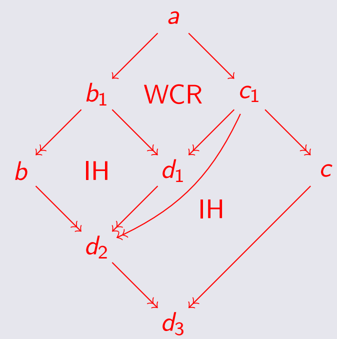
for arbitrary element a

$$\Big(\forall a\colon \big(\forall b\colon a \to b \implies \mathsf{P}(b)\big) \implies \mathsf{P}(a)\Big) \implies \forall a\colon \mathsf{P}(a)$$

Newman's Lemma

SN($\mathcal{A}$) & WCR($\mathcal{A}$) $\implies$ CR($\mathcal{A}$)

First Proof



induction hypothesis CR(b_1) & CR(c_1)

$\forall a'$: if $a \to a'$ then CR(a')

Outline

Definitions (Multiset)

- finite multiset M over A is function from A to $\mathbb{N}$ such that $M(a) = 0$ for all but finitely many $a \in A$
- $M(a)$ is multiplicity of a
- set of all finite multisets over A is denoted by $\mathcal{M}(A)$

Example

$\{a, a, a, b, d, d, d\} \in \mathcal{M}(\{a, b, c, d\})$:

$$a \mapsto 3 \qquad b \mapsto 1 \qquad c \mapsto 0 \qquad d \mapsto 3$$

Definitions (Operations on Multisets)

- sum $\forall a\colon (M_1 \uplus M_2)(a) = M_1(a) + M_2(a)$
- difference $\forall a\colon (M_1 - M_2)(a) = \max\{M_1(a) - M_2(a), 0\}$
- ...

Example

- $\{a, a, a, b, d, d, d\} \uplus \{a, c, c\} = \{a, a, a, a, b, c, c, d, d, d\}$
- $\{a, a, a, b, d, d, d\} - \{a, c, c\} = \{a, a, b, d, d, d\}$

Definition

multiset extension of proper order $>$ on A is relation $>_{\text{mul}}$ defined on $\mathcal{M}(A)$ as follows: $M_1 >_{\text{mul}} M_2$ if $\exists X, Y \in \mathcal{M}(A)$ such that

- $M_2 = (M_1 - X) \uplus Y$
- $\varnothing \neq X \subseteq M_1$
- $\forall y \in Y\ \exists x \in X : x > y$

Example

$$\{2,3\} >_{\text{mul}} \{0,1,3\} >_{\text{mul}} \{0,1,1,2,2,2\} >_{\text{mul}} \{0,1,1,0,1,1,2,2\}$$
$$>_{\text{mul}} \{0,1,0,1,1,2,2\} >_{\text{mul}} \{0,1,0,1,0,0,2\} >_{\text{mul}} \{1,1,1,1,1,1\}$$
$$>_{\text{mul}} \{1,1,1,1\} >_{\text{mul}} \{0,0,0,0,1,1,1\} >_{\text{mul}} \cdots$$

Theorem

- *multiset extension of proper order is proper order*
- *multiset extension of well-founded order is well-founded*

Newman's Lemma

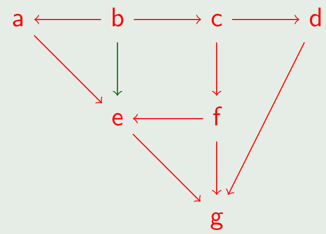
$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(\mathcal{A})$

Second Proof

- given $b\ {}^*\!\leftarrow a \rightarrow^* c$
- construct sequence of conversions $(C_i)_{i \geqslant 0}$ between b and c
  - C_0 is initial conversion $b\ {}^*\!\leftarrow a \rightarrow^* c$
  - C_{i+1} is obtained from C_i by replacing peak $e \leftarrow d \rightarrow f$ in C_i by valley $e \rightarrow^* \cdot\ {}^*\!\leftarrow f$
- $|C_i|$ is multiset of elements appearing in C_i
- $|C_i|\ (\rightarrow^+)_{\text{mul}}\ |C_{i+1}|$
- $(\rightarrow^+)_{\text{mul}}$ is well-founded
- hence $\exists n$ such that C_n has no peaks $\implies C_n : b \rightarrow^* \cdot\ {}^*\!\leftarrow c$

Example

- ARS



- conversion

conversion	multiset	
$a \leftarrow b \rightarrow c \rightarrow d$	$\{a, b, c, d\}$	
$a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$	$\{a, e, f, c, d\}$	
$a \rightarrow e \leftarrow f \rightarrow g \leftarrow d$	$\{a, e, f, g, d\}$	
$a \rightarrow e \rightarrow g \leftarrow d$	$\{a, e, g, d\}$	rewrite proof

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Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems
Gérard Huet
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Proving Termination with Multiset Orderings
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Confluence by Decreasing Diagrams
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