



# Introduction to Term Rewriting

## lecture 2

Aart Middeldorp and Femke van Raamsdonk

Institute of Computer Science  
University of Innsbruck

Department of Computer Science  
VU Amsterdam



Overview

## Outline

- Abstract Rewrite Systems
- Newman's Lemma
- Multiset Orders
- Further Reading

Overview

### Sunday

introduction, examples, **abstract rewriting**, equational reasoning, term rewriting

### Monday

termination, completion

### Tuesday

completion, termination

### Wednesday

confluence, modularity, strategies

### Thursday

exam, advanced topics

AM & FvR

ISR 2010 – lecture 2

2/27

Abstract Rewrite Systems

## Motivation

**concrete** rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- $\lambda$ -calculus
- interaction nets
- ...

**abstract** rewriting

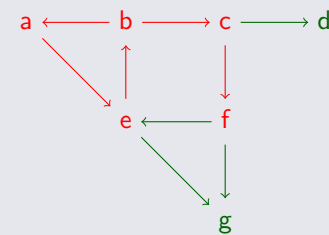
- no structure on objects that are rewritten
- uniform presentation of properties and proofs

## Outline

- Abstract Rewrite Systems
  - Definitions
  - Properties
  - Relationships
- Newman's Lemma
- Multiset Orders
- Further Reading

## Definitions

- **abstract rewrite system (ARS)** is set  $A$  equipped with binary relation  $\rightarrow$



ARS  $\mathcal{A} = \langle A, \rightarrow \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\rightarrow = \left\{ \begin{array}{l} (a, e), (b, a), (b, c), (c, d), (c, f) \\ (e, b), (e, g), (f, e), (f, g) \end{array} \right\}$
- **rewrite sequence**
  - **finite**  $a \rightarrow e \rightarrow b \rightarrow c \rightarrow f$
  - **empty**  $a$
  - **infinite**  $a \rightarrow e \rightarrow b \rightarrow a \rightarrow e \rightarrow b \rightarrow \dots$

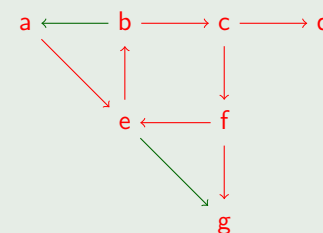
Definition (Derived Relations of  $\rightarrow$ )

- $\leftarrow$  inverse of  $\rightarrow$
- $\rightarrow^*$  transitive and reflexive closure of  $\rightarrow$
- $^*\leftarrow$  inverse of  $\rightarrow^*$  (transitive and reflexive closure of  $\leftarrow$ )
- $\downarrow$  **joinability**  $\downarrow = \rightarrow^* \cdot ^*\leftarrow$
- $\leftrightarrow$  symmetric closure of  $\rightarrow$
- $\leftrightarrow^*$  **conversion** (equivalence relation generated by  $\rightarrow$ )
- $\rightarrow^+$  transitive closure of  $\rightarrow$
- $\rightarrow^=$  reflexive closure of  $\rightarrow$
- $\uparrow$  **meetability**  $\uparrow = ^*\leftarrow \cdot \rightarrow^*$

## Terminology

- if  $x \rightarrow^* y$  then  $x$  **rewrites** to  $y$  and  $y$  is **reduct** of  $x$
- if  $x \rightarrow^* z \ ^*\leftarrow y$  then  $z$  is **common reduct** of  $x$  and  $y$
- if  $x \leftrightarrow^* y$  then  $x$  and  $y$  are **convertible**

## Example

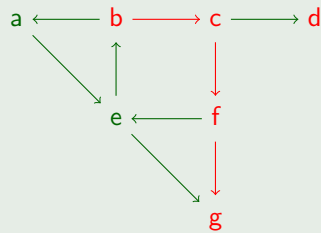


- $a \rightarrow^* f$
- $e \downarrow f$   $f \downarrow d$  **not**  $g \downarrow d$
- $g \leftrightarrow^* d$

### Definition (Normal Forms)

- **normal form** is element  $x$  such that  $x \not\rightarrow y$  for all  $y$
- $NF(\mathcal{A})$  denotes set of normal forms of ARS  $\mathcal{A}$
- $x \rightarrow^! y$  if  $x \rightarrow^* y$  for normal form  $y$  ( $x$  has normal form  $y$ )

### Example



- ARS  $\mathcal{A} = \langle A, \rightarrow \rangle$
- $d$  is normal form
  - $NF(\mathcal{A}) = \{d, g\}$
  - $b \rightarrow^! g$

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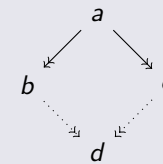
### Definitions

- **SN** strong normalization termination
  - no infinite rewrite sequences
- **WN** weak normalization
  - every element has at least one normal form
  - $\forall a \exists b \ a \rightarrow^! b$
- **UN** unique normal forms
  - no element has more than one normal form
  - $\forall a, b, c \text{ if } a \rightarrow^! b \text{ and } a \rightarrow^! c \text{ then } b = c$
  - $! \leftarrow \cdot \rightarrow^! \subseteq =$

### Definitions

- **CR** confluence Church-Rosser property

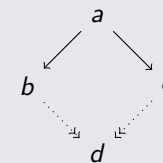
- $\uparrow \subseteq \downarrow$
- $\forall a, b, c$



Wikimedia

- **WCR** local confluence weak Church-Rosser property

- $\leftarrow \cdot \rightarrow \subseteq \downarrow$
- $\forall a, b, c$

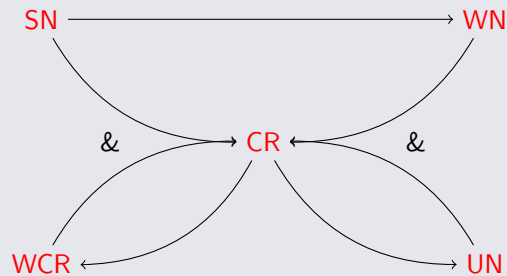


in diagrams:  $\rightarrow$  for  $\rightarrow^*$

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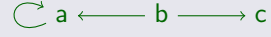

# Summary



# Definitions

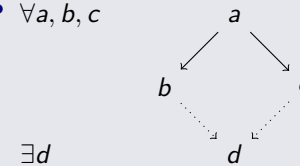
- **semi-completeness**
  - CR & WN
  - every element has unique normal form
- **completeness**
  - CR & SN

# Lemmata

- 1 SN  $\implies$  WN
- 2 SN  $\not\Leftarrow$  WN 
- 3 CR  $\iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow$
- 4 CR  $\implies$  UN
- 5 CR  $\not\Leftarrow$  UN 
- 6 WN & UN  $\implies$  CR
- 7 CR  $\implies$  WCR
- 8 CR  $\not\Leftarrow$  WCR 
- 9 SN & WCR  $\implies$  CR **Newman's Lemma**

# Definition

- **diamond property**  $\diamond$ 
  - $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$



# Lemma

ARS  $\mathcal{A} = \langle A, \rightarrow \rangle$  is confluent if  $\rightarrow \subseteq \rightarrow_{\diamond} \subseteq \rightarrow^*$  for some relation  $\rightarrow_{\diamond}$  on  $A$  with diamond property

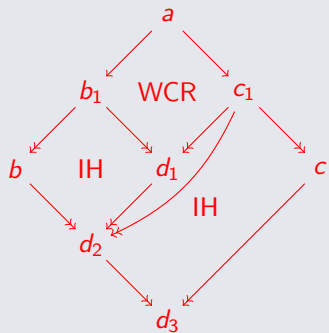
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## Newman's Lemma

$$\text{SN}(\mathcal{A}) \ \& \ \text{WCR}(\mathcal{A}) \ \implies \ \text{CR}(\mathcal{A})$$

## First Proof



induction hypothesis  $\text{CR}(b_1) \ \& \ \text{CR}(c_1)$   
 $\forall a': \text{ if } a \rightarrow a' \text{ then } \text{CR}(a')$

## Well-Founded Induction

given

- property  $P$  of ARSs that satisfies  $P(\mathcal{A}) \iff \forall a: P(a)$
- strongly normalizing ARS  $\mathcal{A} = \langle A, \rightarrow \rangle$

to conclude

- $P(\mathcal{A})$

it is sufficient to prove

- if  $P(b)$  for every  $b$  with  $a \rightarrow b$  then  $P(a)$   
induction hypothesis

for arbitrary element  $a$

$$\left( \forall a: \left( \forall b: a \rightarrow b \implies P(b) \right) \implies P(a) \right) \implies \forall a: P(a)$$

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## Definitions (Multiset)

- finite **multiset**  $M$  over  $A$  is function from  $A$  to  $\mathbb{N}$  such that  $M(a) = 0$  for all but finitely many  $a \in A$
- $M(a)$  is **multiplicity** of  $a$
- set of all finite multisets over  $A$  is denoted by  $\mathcal{M}(A)$

## Example

$\{a, a, a, b, d, d, d\} \in \mathcal{M}(\{a, b, c, d\})$ :

$$a \mapsto 3 \quad b \mapsto 1 \quad c \mapsto 0 \quad d \mapsto 3$$

## Definitions (Operations on Multisets)

- **sum**  $\forall a: (M_1 \uplus M_2)(a) = M_1(a) + M_2(a)$
- **difference**  $\forall a: (M_1 - M_2)(a) = \max\{M_1(a) - M_2(a), 0\}$
- ...

## Example

- $\{a, a, a, b, d, d, d\} \uplus \{a, c, c\} = \{a, a, a, a, b, c, c, d, d, d\}$
- $\{a, a, a, b, d, d, d\} - \{a, c, c\} = \{a, a, b, d, d, d\}$

## Definition

**multiset extension** of proper order  $>$  on  $A$  is relation  $>_{\text{mul}}$  defined on  $\mathcal{M}(A)$  as follows:  $M_1 >_{\text{mul}} M_2$  if  $\exists X, Y \in \mathcal{M}(A)$  such that

- $M_2 = (M_1 - X) \uplus Y$
- $\emptyset \neq X \subseteq M_1$
- $\forall y \in Y \exists x \in X: x > y$

## Example

$$\begin{aligned} \{2, 3\} &>_{\text{mul}} \{0, 1, 3\} >_{\text{mul}} \{0, 1, 1, 2, 2, 2\} >_{\text{mul}} \{0, 1, 1, 0, 1, 1, 2, 2\} \\ &>_{\text{mul}} \{0, 1, 0, 1, 1, 2, 2\} >_{\text{mul}} \{0, 1, 0, 1, 0, 0, 2\} >_{\text{mul}} \{1, 1, 1, 1, 1, 1\} \\ &>_{\text{mul}} \{1, 1, 1, 1\} >_{\text{mul}} \{0, 0, 0, 0, 1, 1, 1\} >_{\text{mul}} \dots \end{aligned}$$

## Theorem

- *multiset extension of proper order is proper order*
- *multiset extension of well-founded order is well-founded*

## Newman's Lemma

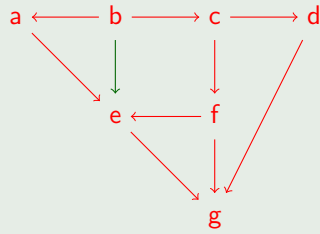
$$\text{SN}(\mathcal{A}) \ \& \ \text{WCR}(\mathcal{A}) \implies \text{CR}(\mathcal{A})$$

## Second Proof

- given  $b \xrightarrow{*} a \xrightarrow{*} c$
- construct sequence of **conversions**  $(C_i)_{i \geq 0}$  between  $b$  and  $c$ 
  - $C_0$  is initial conversion  $b \xrightarrow{*} a \xrightarrow{*} c$
  - $C_{i+1}$  is obtained from  $C_i$  by replacing peak  $e \leftarrow d \rightarrow f$  in  $C_i$  by valley  $e \xrightarrow{*} \cdot \xrightarrow{*} f$
- $|C_i|$  is **multiset** of elements appearing in  $C_i$
- $|C_i| (\rightarrow^+)_{\text{mul}} |C_{i+1}|$
- $(\rightarrow^+)_{\text{mul}}$  is **well-founded**
- hence  $\exists n$  such that  $C_n$  has no peaks  $\implies C_n: b \xrightarrow{*} \cdot \xrightarrow{*} c$

## Example

## • ARS



## • conversion

## multiset

 $a \leftarrow b \rightarrow c \rightarrow d$       $\{a, b, c, d\}$ 
 $a \rightarrow e \leftarrow f \leftarrow c \rightarrow d$       $\{a, e, f, c, d\}$ 
 $a \rightarrow e \leftarrow f \rightarrow g \leftarrow d$       $\{a, e, f, g, d\}$ 
 $a \rightarrow e \rightarrow g \leftarrow d$       $\{a, e, g, d\}$ 

rewrite proof

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## Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems

G rard Huet

JACM 27(4), pp. 797 – 821, 1980

## Proving Termination with Multiset Orderings

Nachum Dershowitz and Zohar Manna

CACM 22(8), pp. 465 – 476, 1979

## Confluence by Decreasing Diagrams

Vincent van Oostrom

TCS 126(2), pp. 259 – 280, 1994