ISR 2010

Introduction to Term Rewriting

lecture 3

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Equational Reasoning
- Algebras
- Term Rewriting
- Further Reading

Definition

equational system (ES) is pair $(\mathcal{F}, \mathcal{E})$ consisting of

- $\mathcal{F}$ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$

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Example

ES $(\mathcal{F}, \mathcal{E})$ with signature $\mathcal{F}$

$$0 \text{ (constant)} \qquad \mathsf{s} \text{ (unary)} \qquad + \text{ (binary, infix)}$$

and equations $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

Inference Rules

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$\mathcal{E} \vdash s \approx t$ ($s \approx_{\mathcal{E}} t$) if equation $s \approx t$ is derivable

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two $\{0, \mathsf{s}, +\}$-algebras

- $\mathcal{A} = (\mathbb{N}, \{0_\mathcal{A}, \mathsf{s}_\mathcal{A}, +_\mathcal{A}\})$ with $0_\mathcal{A} = 0$, $\mathsf{s}_\mathcal{A}(x) = x + 1$, $+_\mathcal{A}(x, y) = x + y$

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interpretation function $[\,\cdot\,]_{\mathcal{A}} \colon \mathcal{T}(\mathcal{F}) \to A$

$$[f(t_1,\ldots,t_n)]_{\mathcal{A}} = f_{\mathcal{A}}([t_1]_{\mathcal{A}},\ldots,[t_n]_{\mathcal{A}})$$

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$$[\alpha]_{\mathcal{B}}(t) = 15 \qquad [\beta]_{\mathcal{B}}(t) = 12$$

Definitions

- equation $s \approx t$ is valid in algebra $\mathcal{A}$ ($\mathcal{A} \vDash s \approx t$, $s =_{\mathcal{A}} t$) if

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  for all assignments α

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- $\mathcal{F}$-algebra $\mathcal{A}$ is model of ES $(\mathcal{F}, \mathcal{E})$ if $s =_{\mathcal{A}} t$ for all equations $s \approx t \in \mathcal{E}$

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- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$
- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{A}$ is model of $\mathcal{E}$

Definitions

- equation $s \approx t$ is valid in algebra $\mathcal{A}$ ($\mathcal{A} \vDash s \approx t$, $s =_{\mathcal{A}} t$) if

$$[\alpha]_{\mathcal{A}}(s) = [\alpha]_{\mathcal{A}}(t)$$

  for all assignments α
- $\mathcal{F}$-algebra $\mathcal{A}$ is model of ES $(\mathcal{F}, \mathcal{E})$ if $s =_{\mathcal{A}} t$ for all equations $s \approx t \in \mathcal{E}$

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$\mathcal{A}$ is model of $\mathcal{E}$ $\qquad$ $\mathcal{B}$ is no model of $\mathcal{E}$

Definition

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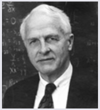
Theorem (Birkhoff)

equational reasoning is sound and complete

$$\forall \text{ ES } \mathcal{E}: \quad \approx_\mathcal{E} \; = \; =_\mathcal{E}$$



Definition

ES $\mathcal{E}$ is consistent if $\exists$ terms s, t such that $s \not\approx_{\mathcal{E}} t$

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Validity Problem

instance: ES $(\mathcal{F}, \mathcal{E})$ terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$
question: $s =_{\mathcal{E}} t$?

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Example (Combinatory Logic)

$$\mathsf{I} \cdot x \approx x$$
$$(\mathsf{K} \cdot x) \cdot y \approx x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$$

Outline

Definition (Positions)

- $\mathcal{P}\text{os}(\,\cdot\,)$

$$\mathcal{P}\text{os}(t) = \begin{cases} \{\epsilon\} & \text{if } t \in \mathcal{V} \\ \{\epsilon\} \cup \{ip \mid p \in \mathcal{P}\text{os}(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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Example

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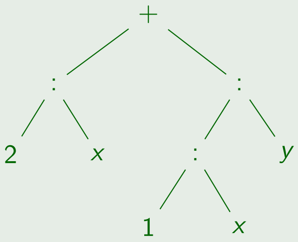

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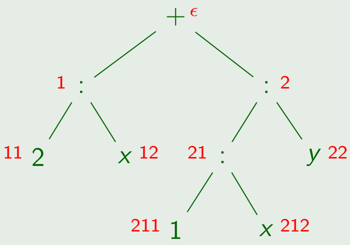

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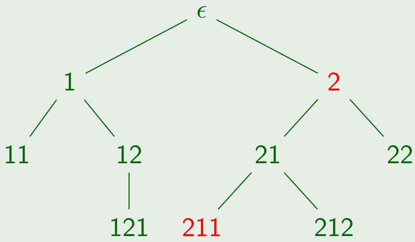
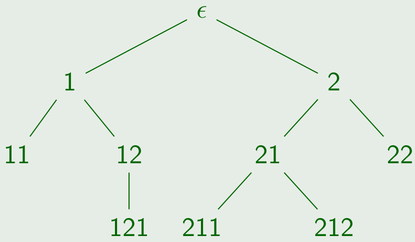


- $2 < 211$

Definition

- $p < q$ if $\exists\, r \neq \epsilon\colon pr = q$ "p is strictly above q" "q is strictly below q"
- $p \leqslant q$ if $\exists\, r\colon pr = q$ "p is above q" "q is below q"

Example



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- $p \parallel q$ if $p \nleqslant q$ and $q \nleqslant p$ "p and q are parallel"

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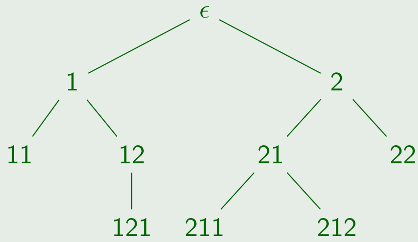

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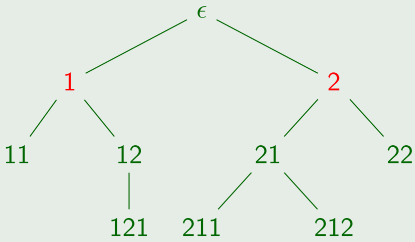

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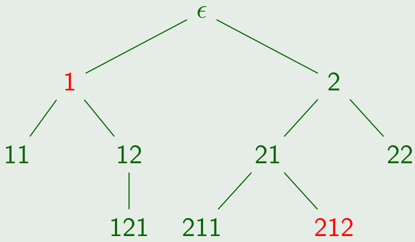

- $2 < 211$
- $1 \parallel 2$
- $1 \parallel 212$

Definitions

- $t|_p$ subterm of t at position p

$$t|_p = \begin{cases} t & \text{if } p = \epsilon \\ t_i|_q & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

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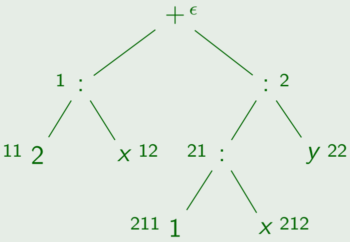
- $t[s]_p$ replace subterm in t at position p by s

$$t[s]_p = \begin{cases} s & \text{if } p = \epsilon \\ f(t_1, \ldots, t_i[s]_q, \ldots, t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

Example

$t = (2 : x) + ((1 : x) : y)$



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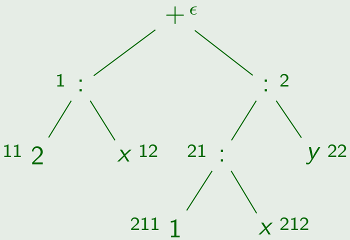

- $t|_{21} = ?$

Example

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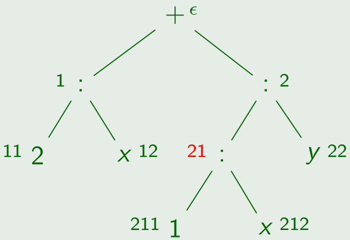

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Example

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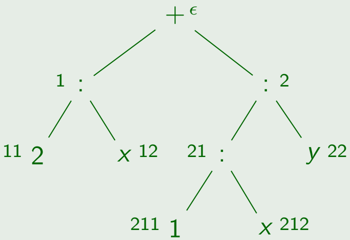

- $t|_{21} = 1 : x$
- $t(212) = ?$

Example

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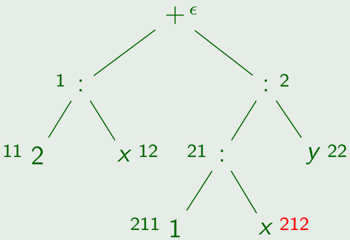

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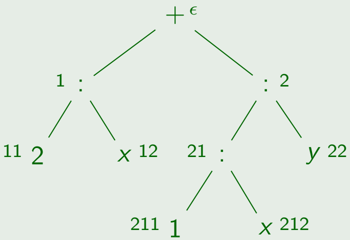
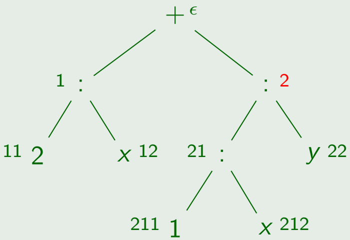


- $t|_{21} = 1 : x$
- $t(212) = x$
- $t[x + 3]_2 = ?$

Example

$t = (2 : x) + ((1 : x) : y)$



- $t|_{21} = 1 : x$
- $t(212) = x$
- $t[x + 3]_2 = (2 : x) + (x + 3)$

Definition

binary relation $\rightarrow_{\mathcal{E}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ for every ES $(\mathcal{F},\mathcal{E})$:

$$s \rightarrow_{\mathcal{E}} t \quad \Longleftrightarrow \quad \begin{array}{l} \exists\ p \in \mathcal{P}\mathsf{os}(s) \\ \exists\ \ell \approx r \in \mathcal{E} \\ \exists\ \text{substitution } \sigma \end{array} \quad \text{with} \quad \begin{array}{rcl} s|_p & = & \ell\sigma \\ t & = & s[r\sigma]_p \end{array}$$

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ES $\mathcal{E} = \{0 + y \approx y,\ \mathsf{s}(x) + y \approx \mathsf{s}(x + y)\}$

$$\begin{array}{c} \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \\ \downarrow_{\mathcal{E}} \\ \mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0))) \end{array}$$

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	position	equation	substitution
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$\downarrow_\mathcal{E}$	1	$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$	$\{x \mapsto 0, y \mapsto \mathsf{s}(0)\}$
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Lemma

$\rightarrow_{\mathcal{E}}$ *is smallest relation that contains* $\mathcal{E}$ *and is closed under contexts and substitutions*

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with every ES $(\mathcal{F}, \mathcal{E})$ we associate ARS $\langle \mathcal{T}(\mathcal{F}, \mathcal{V}), \rightarrow_{\mathcal{E}} \rangle$

- notation ($\rightarrow^*_{\mathcal{E}}$, $\leftrightarrow^*_{\mathcal{E}}$, NF($\mathcal{E}$), ...)
- properties (SN, CR, ...)

are obtained for free

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Theorem

$\forall$ *ES* $\mathcal{E}$ $\quad \leftrightarrow^*_{\mathcal{E}} \ = \ \approx_{\mathcal{E}} \ = \ =_{\mathcal{E}}$

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$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))}[a] \qquad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}[a]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}[c]}{\dfrac{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}[c]}[t]$$

$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \leftrightarrow^*_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \to_{\mathcal{E}} \mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0))) \to_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$$

Definitions

- rewrite rule $(\ell \to r)$ is equation $\ell \approx r$ such that
  - $\ell \notin \mathcal{V}$
  - $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}(\ell)$

Definitions

- rewrite rule $(\ell \to r)$ is equation $\ell \approx r$ such that
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  - $\mathcal{F}$ signature
  - $\mathcal{R}$ set of rewrite rules between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$

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Lemma

TRSs are ESs

Validity Problem

instance: ES $\mathcal{E}$ terms s, t

question: $s =_{\mathcal{E}} t$?

Theorem

validity problem is undecidable

Validity Problem

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question: $s =_{\mathcal{E}} t$?

Theorem

validity problem is *decidable* *for ES $\mathcal{E}$ if $\exists$ finite TRS $\mathcal{R}$ such that*

1. *$\mathcal{R}$ is complete (confluent and terminating)*

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validity problem is *decidable* *for ES $\mathcal{E}$ if $\exists$ finite TRS $\mathcal{R}$ such that*

1. *$\mathcal{R}$ is complete (confluent and terminating)*
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

Example (Group Theory)

signature	e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)	
ES	$e \cdot x \approx x \qquad x^- \cdot x \approx e \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$	$\mathcal{E}$
theorems	$e^- \approx_{\mathcal{E}} e \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$	
TRS	see below	$\mathcal{R}$

$$
\begin{array}{rcl@{\qquad}rcl}
e \cdot x & \to & x & x \cdot e & \to & x \\
x^- \cdot x & \to & e & x \cdot x^- & \to & e \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
e^- & \to & e & (x \cdot y)^- & \to & y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to & y & x \cdot (x^- \cdot y) & \to & y
\end{array}
$$

Example (Group Theory)

signature	e (constant)	$^-$ (unary, postfix)	$\cdot$ (binary, infix)
ES	$e \cdot x \approx x \qquad x^- \cdot x \approx e \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$		$\mathcal{E}$
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TRS $\qquad\qquad \mathcal{R}$

$$
\begin{array}{rclcrcl}
e \cdot x & \to & x & \qquad & x \cdot e & \to & x \\
x^- \cdot x & \to & e & & x \cdot x^- & \to & e \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & & x^{--} & \to & x \\
e^- & \to & e & & (x \cdot y)^- & \to & y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to & y & & x \cdot (x^- \cdot y) & \to & y
\end{array}
$$

- $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} \implies \mathcal{E}$ has decidable validity problem

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ES: $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems: $\mathsf{e}^- \downarrow_{\mathcal{R}} \mathsf{e} \qquad (x \cdot y)^- \downarrow_{\mathcal{R}} y^- \cdot z^-$

TRS: $\mathcal{R}$

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

- $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} \implies \mathcal{E}$ has decidable validity problem

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signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

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TRS

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned} \qquad \mathcal{R}$$

- $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} \implies \mathcal{E}$ has decidable validity problem
- how to compute $\mathcal{R}$?

Example (Group Theory)

signature $\quad \mathsf{e}$ (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)

ES $\quad \mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\quad \mathsf{e}^{-} \downarrow_{\mathcal{R}} \mathsf{e} \qquad (x \cdot y)^{-} \downarrow_{\mathcal{R}} y^{-} \cdot z^{-}$

TRS $\qquad \mathcal{R}$

$$
\begin{array}{rclrcl}
\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\
x^{-} \cdot x & \to & \mathsf{e} & x \cdot x^{-} & \to & \mathsf{e} \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
\mathsf{e}^{-} & \to & \mathsf{e} & (x \cdot y)^{-} & \to & y^{-} \cdot x^{-} \\
x^{-} \cdot (x \cdot y) & \to & y & x \cdot (x^{-} \cdot y) & \to & y
\end{array}
$$

- $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} \implies \mathcal{E}$ has decidable validity problem
- how to compute $\mathcal{R}$? completion (lectures 5 & 6)

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ES $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

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TRS

$$\begin{array}{rclrcl} \mathsf{e} \cdot x & \rightarrow & x & x \cdot \mathsf{e} & \rightarrow & x \\ x^- \cdot x & \rightarrow & \mathsf{e} & x \cdot x^- & \rightarrow & \mathsf{e} \\ (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) & x^{--} & \rightarrow & x \\ \mathsf{e}^- & \rightarrow & \mathsf{e} & (x \cdot y)^- & \rightarrow & y^- \cdot x^- \\ x^- \cdot (x \cdot y) & \rightarrow & y & x \cdot (x^- \cdot y) & \rightarrow & y \end{array} \qquad \mathcal{R}$$

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(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & & x^{--} & \to & x \\
\mathsf{e}^- & \to & \mathsf{e} & & (x \cdot y)^- & \to & y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to & y & & x \cdot (x^- \cdot y) & \to & y
\end{array}
$$

- $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} \quad \Longrightarrow \quad \mathcal{E}$ has decidable validity problem
- how to compute $\mathcal{R}$? completion (lectures 5 & 6)
- how to prove termination of $\mathcal{R}$? LPO or KBO (lectures 4 & 7)

Example

TRS $\mathcal{R}$ modeling Sieve of Eratostheness for generating list of prime numbers

$$
\begin{aligned}
\mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y) \\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x)) \\
\mathsf{head}(x : y) &\to x & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w) \\
\mathsf{tail}(x : y) &\to y & \mathsf{filter}(\mathsf{s}(x), y : z, w) &\to y : \mathsf{filter}(x, z, w)
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\end{aligned}
$$

- $\mathcal{R}$ is confluent but not terminating

$$\mathsf{from}(0) \to 0 : \mathsf{from}(\mathsf{s}(0)) \to 0 : (\mathsf{s}(0) : \mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \to \cdots$$

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- how to prove confluence of $\mathcal{R}$?

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- how to prove confluence of $\mathcal{R}$? orthogonality (lecture 8)

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- how to prove confluence of $\mathcal{R}$? orthogonality (lecture 8)
- $\exists$ non-terminating terms with (unique) normal form

$$\mathsf{head}(\mathsf{tail}(\mathsf{tail}(\mathsf{primes}))) \to^! \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))))$$

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$$\mathsf{from}(0) \to 0 : \mathsf{from}(\mathsf{s}(0)) \to 0 : (\mathsf{s}(0) : \mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \to \cdots$$

- how to prove confluence of $\mathcal{R}$? orthogonality (lecture 8)
- $\exists$ non-terminating terms with (unique) normal form

$$\mathsf{head}(\mathsf{tail}(\mathsf{tail}(\mathsf{primes}))) \to^{!} \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))))$$

- how to compute normal forms in $\mathcal{R}$? strategy (lecture 9)

Example (Combinatory Logic)

$$
\begin{aligned}
\mathsf{I} \cdot x &\approx x \\
(\mathsf{K} \cdot x) \cdot y &\approx x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z &\approx (x \cdot z) \cdot (y \cdot z)
\end{aligned}
$$

Example (Combinatory Logic)

$$
\begin{aligned}
\mathsf{I} \cdot x &\to x \\
(\mathsf{K} \cdot x) \cdot y &\to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z)
\end{aligned}
$$

Example (Combinatory Logic)

$$
\begin{aligned}
\mathsf{I} \cdot x &\to x & \mathsf{I}\, x &\to x \\
(\mathsf{K} \cdot x) \cdot y &\to x & (\mathsf{K}\, x)\, y &\to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\, x)\, y)\, z &\to (x\, z)\, (y\, z)
\end{aligned}
$$

- applicative notation: surpress $\cdot$

Example (Combinatory Logic)

$$
\begin{array}{rlrlrl}
\mathsf{I} \cdot x & \rightarrow x & \mathsf{I}\, x & \rightarrow x & \mathsf{I}\, x & \rightarrow x \\
(\mathsf{K} \cdot x) \cdot y & \rightarrow x & (\mathsf{K}\, x)\, y & \rightarrow x & \mathsf{K}\, x\, y & \rightarrow x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z & \rightarrow (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\, x)\, y)\, z & \rightarrow (x\, z)\, (y\, z) & \mathsf{S}\, x\, y\, z & \rightarrow x\, z\, (y\, z)
\end{array}
$$

- applicative notation: surpress $\cdot$ and adopt left-association

Example (Combinatory Logic)

$$
\begin{aligned}
\mathsf{I} \cdot x &\to x & \mathsf{I}\,x &\to x & \mathsf{I}\,x &\to x \\
(\mathsf{K} \cdot x) \cdot y &\to x & (\mathsf{K}\,x)\,y &\to x & \mathsf{K}\,x\,y &\to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\,x)\,y)\,z &\to (x\,z)\,(y\,z) & \mathsf{S}\,x\,y\,z &\to x\,z\,(y\,z)
\end{aligned}
$$

- applicative notation: surpress $\cdot$ and adopt left-association
- CL is confluent but not terminating

$$\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$$

Example (Combinatory Logic)

$$\begin{array}{rlrlrl}
\mathsf{I}\cdot x & \to x & \mathsf{I}\,x & \to x & \mathsf{I}\,x & \to x \\
(\mathsf{K}\cdot x)\cdot y & \to x & (\mathsf{K}\,x)\,y & \to x & \mathsf{K}\,x\,y & \to x \\
((\mathsf{S}\cdot x)\cdot y)\cdot z & \to (x\cdot z)\cdot(y\cdot z) & ((\mathsf{S}\,x)\,y)\,z & \to (x\,z)\,(y\,z) & \mathsf{S}\,x\,y\,z & \to x\,z\,(y\,z)
\end{array}$$

- applicative notation: surpress $\cdot$ and adopt left-association
- CL is confluent but not terminating

$$\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$$

- CL is consistent

$$\mathsf{S} \not\leftrightarrow^* \mathsf{K}$$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is string rewrite system (SRS) if $\mathcal{F}$ consists of unary function symbols

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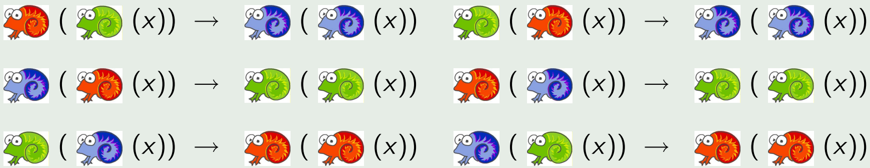
Example



Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is string rewrite system (SRS) if $\mathcal{F}$ consists of unary function symbols

Example

Theorem

term rewriting is Turing-complete hence all non-trivial questions are undecidable

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Undecidable Problems

instance:	(finite) TRS $\mathcal{R}$
question:	is $\mathcal{R}$ terminating ?

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Undecidable Problems

instance:	(finite) TRS $\mathcal{R}$
question:	is $\mathcal{R}$ terminating ?

instance:	TRS $\mathcal{R}$
question:	is $\mathcal{R}$ confluent ?

Theorem

term rewriting is Turing-complete hence all non-trivial questions are undecidable

Undecidable Problems

instance:	(finite) TRS $\mathcal{R}$
question:	is $\mathcal{R}$ terminating ?

instance:	TRS $\mathcal{R}$
question:	is $\mathcal{R}$ confluent ?

instance:	TRS $\mathcal{R}$ term t
question:	is t terminating ?

Theorem

term rewriting is Turing-complete hence all non-trivial questions are undecidable

Undecidable Problems

instance: (finite) TRS $\mathcal{R}$
question: is $\mathcal{R}$ terminating ?

instance: TRS $\mathcal{R}$
question: is $\mathcal{R}$ confluent ?

instance: TRS $\mathcal{R}$ term t
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...

Theorem

term rewriting is Turing-complete hence all non-trivial questions are undecidable

Undecidable Problems

instance: (finite) TRS $\mathcal{R}$
question: is $\mathcal{R}$ terminating ?

instance: TRS $\mathcal{R}$
question: is $\mathcal{R}$ confluent ?

instance: TRS $\mathcal{R}$ term t
question: is t terminating ?

...

Theorem

- *confluence is decidable for terminating TRSs*

Theorem

term rewriting is Turing-complete hence all non-trivial questions are undecidable

Undecidable Problems

instance: (finite) TRS $\mathcal{R}$
question: is $\mathcal{R}$ terminating ?

instance: TRS $\mathcal{R}$
question: is $\mathcal{R}$ confluent ?

instance: TRS $\mathcal{R}$ term t
question: is t terminating ?

...

Theorem

- *confluence is decidable for terminating TRSs*
- *termination is undecidable for confluent TRSs*

Theorem

most problems for ground TRSs are decidable

Theorem

most problems for ground TRSs are decidable

Definitions

- rewrite rule $\ell \longrightarrow r$ is right-ground if r is ground

Theorem

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Definitions

- rewrite rule $\ell \longrightarrow r$ is right-ground if r is ground
- rewrite rule $\ell \longrightarrow r$ is ground if ℓ and r are ground

Theorem

most problems for ground TRSs are decidable

Definitions

- rewrite rule $\ell \to r$ is right-ground if r is ground
- rewrite rule $\ell \to r$ is ground if ℓ and r are ground
- TRS is (right-)ground if all rewrite rules are (right-)ground

Theorem

most problems for ground TRSs are decidable

Definitions

- rewrite rule $\ell \to r$ is right-ground if r is ground
- rewrite rule $\ell \to r$ is ground if ℓ and r are ground
- TRS is (right-)ground if all rewrite rules are (right-)ground

Theorem

validity problem is decidable for ground ESs

Theorem

most problems for ground TRSs are decidable

Definitions

- rewrite rule $\ell \rightarrow r$ is right-ground if r is ground
- rewrite rule $\ell \rightarrow r$ is ground if ℓ and r are ground
- TRS is (right-)ground if all rewrite rules are (right-)ground

Theorem

validity problem is decidable for ground ESs *congruence closure*

Outline

On the Uniform Halting Problem for Term Rewriting Systems
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