ISR 2010

Introduction to Term Rewriting

lecture 3

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Equational Reasoning
- Algebras
- Term Rewriting
- Further Reading

Definition

equational system (ES) is pair $(\mathcal{F}, \mathcal{E})$ consisting of

- $\mathcal{F}$ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$

Example

ES $(\mathcal{F}, \mathcal{E})$ with signature $\mathcal{F}$

$$0 \text{ (constant)} \qquad \mathsf{s} \text{ (unary)} \qquad + \text{ (binary, infix)}$$

and equations $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

Inference Rules

[r]	reflexivity	$\dfrac{}{t \approx t}$	$\forall\ t$
[s]	symmetry	$\dfrac{s \approx t}{t \approx s}$	
[t]	transitivity	$\dfrac{s \approx t,\ t \approx u}{s \approx u}$	
[a]	application	$\dfrac{}{l\sigma \approx r\sigma}$	$\forall\ l \approx r \in \mathcal{E}\ \forall\ \sigma$
[c]	congruence	$\dfrac{s_1 \approx t_1,\ \ldots,\ s_n \approx t_n}{f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n)}$	$\forall$ n-ary f

Definition

$\mathcal{E} \vdash s \approx t$ ($s \approx_{\mathcal{E}} t$) if equation $s \approx t$ is derivable

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{\dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))}\,[\mathrm{a}] \qquad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\,[\mathrm{a}]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}\,[\mathrm{c}]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}\,[\mathrm{t}]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\,[\mathrm{c}]$$

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Definition

$\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ consists of

- carrier A
- interpretations $f_{\mathcal{A}} \colon \underbrace{A \times \cdots \times A}_{n} \to A$ if $f \in \mathcal{F}$ has arity n

Example

two $\{0, \mathsf{s}, +\}$-algebras

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$

Definition

interpretation function $[\cdot]_{\mathcal{A}} \colon \mathcal{T}(\mathcal{F}) \to A$

$$[f(t_1, \ldots, t_n)]_{\mathcal{A}} = f_{\mathcal{A}}([t_1]_{\mathcal{A}}, \ldots, [t_n]_{\mathcal{A}})$$

Example

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$

$$[\,\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))\,]_{\mathcal{A}} = 4 \qquad [\,\mathsf{s}(\mathsf{s}(0)) + \mathsf{s}(0 + \mathsf{s}(0))\,]_{\mathcal{A}} = 4$$

$$[\,\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))\,]_{\mathcal{B}} = 5 \qquad [\,\mathsf{s}(\mathsf{s}(0)) + \mathsf{s}(0 + \mathsf{s}(0))\,]_{\mathcal{B}} = 11$$

Definitions

- assignment $\alpha\colon \mathcal{V} \to A \quad (\alpha \in A^{\mathcal{V}})$
- interpretation function $[\alpha]_{\mathcal{A}}(\,\cdot\,)\colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to A$

$$[\alpha]_{\mathcal{A}}(t) = \begin{cases} \alpha(t) & \text{if } t \in \mathcal{V} \\ f_{\mathcal{A}}([\alpha]_{\mathcal{A}}(t_1),\dots,[\alpha]_{\mathcal{A}}(t_n)) & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

Example

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x,y) = x + y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x,y) = 2x + y$
- $t = \mathsf{s}(\mathsf{s}(x) + \mathsf{s}(x + y)) \quad \alpha(x) = 2 \quad \alpha(y) = 3 \quad \beta(x) = 1 \quad \beta(y) = 4$

$$[\alpha]_{\mathcal{A}}(t) = 10 \qquad [\beta]_{\mathcal{A}}(t) = 9$$

$$[\alpha]_{\mathcal{B}}(t) = 15 \qquad [\beta]_{\mathcal{B}}(t) = 12$$

Definitions

- equation $s \approx t$ is valid in algebra $\mathcal{A}$ ($\mathcal{A} \vDash s \approx t$, $s =_{\mathcal{A}} t$) if

$$[\alpha]_{\mathcal{A}}(s) = [\alpha]_{\mathcal{A}}(t)$$

  for all assignments α

- $\mathcal{F}$-algebra $\mathcal{A}$ is model of ES $(\mathcal{F}, \mathcal{E})$ if $s =_{\mathcal{A}} t$ for all equations $s \approx t \in \mathcal{E}$

Example

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$
- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{A}$ is model of $\mathcal{E}$ $\qquad$ $\mathcal{B}$ is no model of $\mathcal{E}$

Definition

- $\mathcal{E} \vDash s \approx t$ ($s =_{\mathcal{E}} t$) if equation $s \approx t$ is valid in all models of $\mathcal{E}$
- equational theory of $\mathcal{E}$ consists of all equations $s \approx t$ such that $\mathcal{E} \vDash s \approx t$

Example

- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

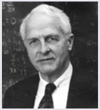
- model $\mathcal{C} = (\mathbb{N}, \{0_{\mathcal{C}}, \mathsf{s}_{\mathcal{C}}, +_{\mathcal{C}}\})$ with $0_{\mathcal{C}} = 0$, $\mathsf{s}_{\mathcal{C}}(x) = x$, $+_{\mathcal{C}}(x, y) = y$

$$\mathcal{E} \vDash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \qquad \mathcal{E} \nvDash x + y \approx y + x$$

Theorem (Birkhoff)

equational reasoning is sound and complete

$$\forall \textit{ ES } \mathcal{E}: \quad \approx_{\mathcal{E}} \; = \; =_{\mathcal{E}}$$



Definition

ES $\mathcal{E}$ is consistent if $\exists$ terms s, t such that $s \not\approx_{\mathcal{E}} t$

Validity Problem

instance: ES $(\mathcal{F}, \mathcal{E})$ terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$
question: $s =_{\mathcal{E}} t$?

Theorem

validity problem is undecidable

Example (Combinatory Logic)

$$\begin{aligned} \mathsf{I} \cdot x &\approx x \\ (\mathsf{K} \cdot x) \cdot y &\approx x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\approx (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

Outline

- Equational Reasoning
- Algebras
- **Term Rewriting**
- Further Reading

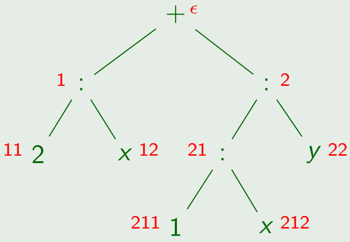
Definition (Positions)

- $\mathcal{P}\mathrm{os}(\,\cdot\,)$

$$\mathcal{P}\mathrm{os}(t) = \begin{cases} \{\epsilon\} & \text{if } t \in \mathcal{V} \\ \{\epsilon\} \cup \{ip \mid p \in \mathcal{P}\mathrm{os}(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Example

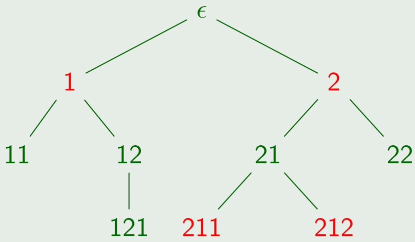
$(2 : x) + ((1 : x) : y)$



Definition

- $p < q$ if $\exists\, r \neq \epsilon\colon pr = q$ "p is strictly above q" "q is strictly below q"
- $p \leqslant q$ if $\exists\, r\colon pr = q$ "p is above q" "q is below q"
- $p \parallel q$ if $p \nleqslant q$ and $q \nleqslant p$ "p and q are parallel"

Example



- $2 < 211$
- $1 \parallel 2$
- $1 \parallel 212$

Definitions

- $t|_p$ subterm of t at position p

$$t|_p = \begin{cases} t & \text{if } p = \epsilon \\ t_i|_q & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

- $t(p)$ symbol in t at position p

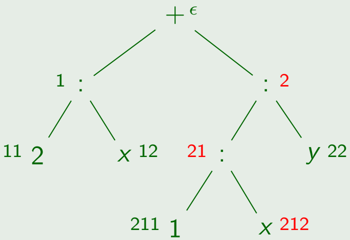
$$t(p) = \begin{cases} \text{root}(t) & \text{if } p = \epsilon \\ t_i(q) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

- $t[s]_p$ replace subterm in t at position p by s

$$t[s]_p = \begin{cases} s & \text{if } p = \epsilon \\ f(t_1, \ldots, t_i[s]_q, \ldots, t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

Example

$t = (2 : x) + ((1 : x) : y)$



- $t|_{21} = 1 : x$
- $t(212) = x$
- $t[x + 3]_2 = (2 : x) + (x + 3)$

Definition

binary relation $\rightarrow_{\mathcal{E}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ for every ES $(\mathcal{F},\mathcal{E})$:

$$s \rightarrow_{\mathcal{E}} t \quad \Longleftrightarrow \quad \begin{array}{l} \exists\ p \in \mathcal{P}\mathsf{os}(s) \\ \exists\ \ell \approx r \in \mathcal{E} \\ \exists \text{ substitution } \sigma \end{array} \quad \text{with} \quad \begin{array}{rcl} s|_p & = & \ell\sigma \\ t & = & s[r\sigma]_p \end{array} \qquad \text{redex}$$

Example

ES $\mathcal{E} = \{0 + y \approx y,\ \mathsf{s}(x) + y \approx \mathsf{s}(x + y)\}$

	position	equation	substitution
$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0))$			
$\downarrow_{\mathcal{E}}$	1	$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$	$\{x \mapsto 0, y \mapsto \mathsf{s}(0)\}$
$\mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0)))$			
$\downarrow_{\mathcal{E}}$	11	$0 + y \approx y$	$\{y \mapsto \mathsf{s}(0)\}$
$\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$			

Lemma

$\rightarrow_{\mathcal{E}}$ is smallest relation that contains $\mathcal{E}$ and is closed under contexts and substitutions

Remark

with every ES $(\mathcal{F}, \mathcal{E})$ we associate ARS $\langle \mathcal{T}(\mathcal{F}, \mathcal{V}), \rightarrow_{\mathcal{E}} \rangle$

- notation ($\rightarrow_{\mathcal{E}}^*$, $\leftrightarrow_{\mathcal{E}}^*$, NF($\mathcal{E}$), ...)
- properties (SN, CR, ...)

are obtained for free

Theorem

$\forall$ *ES* $\mathcal{E}$ $\quad \leftrightarrow_{\mathcal{E}}^* \;=\; \approx_{\mathcal{E}} \;=\; =_{\mathcal{E}}$

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{\dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))}\,[\mathrm{a}] \qquad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\,[\mathrm{a}]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}\,[\mathrm{c}]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}\,[\mathrm{t}]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\,[\mathrm{c}]$$

$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \leftrightarrow^*_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \to_{\mathcal{E}} \mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0))) \to_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$$

Definitions

- rewrite rule ($\ell \to r$) is equation $\ell \approx r$ such that
  - $\ell \notin \mathcal{V}$
  - $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}(\ell)$
- term rewrite system (TRS) is pair $(\mathcal{F}, \mathcal{R})$ consisting of
  - $\mathcal{F}$ signature
  - $\mathcal{R}$ set of rewrite rules between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$

Lemma

TRSs are ESs

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $s =_{\mathcal{E}} t$?

Theorem

validity problem is decidable for ES $\mathcal{E}$ if $\exists$ finite TRS $\mathcal{R}$ such that

1. *$\mathcal{R}$ is complete (confluent and terminating)*
2. $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$

Example (Group Theory)

signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

ES $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^- \downarrow_{\mathcal{R}} \mathsf{e} \qquad (x \cdot y)^- \downarrow_{\mathcal{R}} y^- \cdot z^-$

TRS

$$\begin{aligned}
\mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\
x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\
(x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\
\mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\
x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y
\end{aligned} \qquad \mathcal{R}$$

- $\mathcal{R}$ is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*} \implies \mathcal{E}$ has decidable validity problem
- how to compute $\mathcal{R}$? completion (lectures 5 & 6)
- how to prove termination of $\mathcal{R}$? LPO or KBO (lectures 4 & 7)

Example

TRS $\mathcal{R}$ modeling Sieve of Eratostheness for generating list of prime numbers

$$\begin{aligned}
\mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y) \\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x)) \\
\mathsf{head}(x : y) &\to x & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w) \\
\mathsf{tail}(x : y) &\to y & \mathsf{filter}(\mathsf{s}(x), y : z, w) &\to y : \mathsf{filter}(x, z, w)
\end{aligned}$$

- $\mathcal{R}$ is confluent but not terminating

$$\mathsf{from}(0) \to 0 : \mathsf{from}(\mathsf{s}(0)) \to 0 : (\mathsf{s}(0) : \mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \to \cdots$$

- how to prove confluence of $\mathcal{R}$? orthogonality (lecture 8)
- $\exists$ non-terminating terms with (unique) normal form

$$\mathsf{head}(\mathsf{tail}(\mathsf{tail}(\mathsf{primes}))) \to^! \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))))$$

- how to compute normal forms in $\mathcal{R}$? strategy (lecture 9)

Example (Combinatory Logic)

$$
\begin{array}{rlrlrl}
\mathsf{I} \cdot x & \to x & \mathsf{I}\, x & \to x & \mathsf{I}\, x & \to x \\
(\mathsf{K} \cdot x) \cdot y & \to x & (\mathsf{K}\, x)\, y & \to x & \mathsf{K}\, x\, y & \to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z & \to (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\, x)\, y)\, z & \to (x\, z)\, (y\, z) & \mathsf{S}\, x\, y\, z & \to x\, z\, (y\, z)
\end{array}
$$

- applicative notation: surpress $\cdot$ and adopt left-association
- CL is confluent but not terminating

$$\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$$

- CL is consistent

$$\mathsf{S} \not\leftrightarrow^* \mathsf{K}$$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is string rewrite system (SRS) if $\mathcal{F}$ consists of unary function symbols

Example

Theorem

term rewriting is Turing-complete hence all non-trivial questions are undecidable

Undecidable Problems

instance: (finite) TRS $\mathcal{R}$
question: is $\mathcal{R}$ terminating ?

instance: TRS $\mathcal{R}$
question: is $\mathcal{R}$ confluent ?

instance: TRS $\mathcal{R}$ term t
question: is t terminating ?

...

Theorem

- *confluence is decidable for terminating TRSs*
- *termination is undecidable for confluent TRSs*

Theorem

most problems for ground TRSs are decidable

Definitions

- rewrite rule $\ell \rightarrow r$ is right-ground if r is ground
- rewrite rule $\ell \rightarrow r$ is ground if ℓ and r are ground
- TRS is (right-)ground if all rewrite rules are (right-)ground

Theorem

validity problem is decidable for ground ESs *congruence closure*

Outline

On the Uniform Halting Problem for Term Rewriting Systems
Gérard Huet and Dallas Lankford
technical report 283, INRIA, 1978

Termination of Linear Rewriting Systems (Preliminary Version)
Nachum Dershowitz
Proc. 8th ICALP, pp. 448–458, 1981

Fast Congruence Closure and Extensions
Robert Nieuwenhuis and Albert Oliveras
I&C 205(4), pp. 557–580, 2007