Introduction to Term Rewriting

lecture 3

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Equational Reasoning
- Algebras
- Term Rewriting
- Further Reading

Definition

equational system (ES) is pair $(\mathcal{F}, \mathcal{E})$ consisting of

- $\mathcal{F}$ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$

Example

ES $(\mathcal{F}, \mathcal{E})$ with signature $\mathcal{F}$

$\mathsf{0}$ (constant) s (unary) $+$ (binary, infix)

and equations $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

Inference Rules

[r]	reflexivity	$\dfrac{}{t \approx t}$	$\forall\ t$
[s]	symmetry	$\dfrac{s \approx t}{t \approx s}$	
[t]	transitivity	$\dfrac{s \approx t,\ t \approx u}{s \approx u}$	
[a]	application	$\dfrac{}{l\sigma \approx r\sigma}$	$\forall\ l \approx r \in \mathcal{E}\ \forall\ \sigma$
[c]	congruence	$\dfrac{s_1 \approx t_1,\ \ldots,\ s_n \approx t_n}{f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n)}$	$\forall$ n-ary f

Definition

$\mathcal{E} \vdash s \approx t$ ($s \approx_{\mathcal{E}} t$) if equation $s \approx t$ is derivable

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{[a]\ \dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))} \qquad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\ [a]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}\ [c]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}\ [t]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\ [c]$$

Outline

Definition

$\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ consists of

- carrier A
- interpretations $f_{\mathcal{A}} \colon \underbrace{A \times \cdots \times A}_{n} \to A$ if $f \in \mathcal{F}$ has arity n

Example

two $\{0, \mathsf{s}, +\}$-algebras

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$

Definition

interpretation function $[\,\cdot\,]_{\mathcal{A}} \colon \mathcal{T}(\mathcal{F}) \to A$

$$[f(t_1,\ldots,t_n)]_{\mathcal{A}} = f_{\mathcal{A}}([t_1]_{\mathcal{A}},\ldots,[t_n]_{\mathcal{A}})$$

Example

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x+1$, $+_{\mathcal{A}}(x,y) = x+y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x+1$, $+_{\mathcal{B}}(x,y) = 2x+y$

$$[\,\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))\,]_{\mathcal{A}} = 4 \qquad [\,\mathsf{s}(\mathsf{s}(0)) + \mathsf{s}(0+\mathsf{s}(0))\,]_{\mathcal{A}} = 4$$

$$[\,\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))\,]_{\mathcal{B}} = 5 \qquad [\,\mathsf{s}(\mathsf{s}(0)) + \mathsf{s}(0+\mathsf{s}(0))\,]_{\mathcal{B}} = 11$$

Definitions

- assignment $\alpha \colon \mathcal{V} \to A \quad (\alpha \in A^{\mathcal{V}})$
- interpretation function $[\alpha]_{\mathcal{A}}(\,\cdot\,) \colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to A$

$$[\alpha]_{\mathcal{A}}(t) = \begin{cases} \alpha(t) & \text{if } t \in \mathcal{V} \\ f_{\mathcal{A}}([\alpha]_{\mathcal{A}}(t_1),\ldots,[\alpha]_{\mathcal{A}}(t_n)) & \text{if } t = f(t_1,\ldots,t_n) \end{cases}$$

Example

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x+1$, $+_{\mathcal{A}}(x,y) = x+y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x+1$, $+_{\mathcal{B}}(x,y) = 2x+y$
- $t = \mathsf{s}(\mathsf{s}(x) + \mathsf{s}(x+y))$ $\quad \alpha(x) = 2 \quad \alpha(y) = 3 \quad \beta(x) = 1 \quad \beta(y) = 4$

$$[\alpha]_{\mathcal{A}}(t) = 10 \qquad [\beta]_{\mathcal{A}}(t) = 9$$

$$[\alpha]_{\mathcal{B}}(t) = 15 \qquad [\beta]_{\mathcal{B}}(t) = 12$$

Definitions

- equation $s \approx t$ is valid in algebra $\mathcal{A}$ ($\mathcal{A} \vDash s \approx t$, $s =_{\mathcal{A}} t$) if

$$[\alpha]_{\mathcal{A}}(s) = [\alpha]_{\mathcal{A}}(t)$$

  for all assignments α
- $\mathcal{F}$-algebra $\mathcal{A}$ is model of ES $(\mathcal{F}, \mathcal{E})$ if $s =_{\mathcal{A}} t$ for all equations $s \approx t \in \mathcal{E}$

Example

- $\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, \mathsf{s}_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $\mathsf{s}_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, \mathsf{s}_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $\mathsf{s}_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$
- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{A}$ is model of $\mathcal{E}$ $\qquad$ $\mathcal{B}$ is no model of $\mathcal{E}$

Definition

- $\mathcal{E} \vDash s \approx t$ ($s =_{\mathcal{E}} t$) if equation $s \approx t$ is valid in all models of $\mathcal{E}$
- equational theory of $\mathcal{E}$ consists of all equations $s \approx t$ such that $\mathcal{E} \vDash s \approx t$

Example

- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

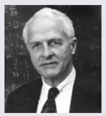
- model $\mathcal{C} = (\mathbb{N}, \{0_{\mathcal{C}}, \mathsf{s}_{\mathcal{C}}, +_{\mathcal{C}}\})$ with $0_{\mathcal{C}} = 0$, $\mathsf{s}_{\mathcal{C}}(x) = x$, $+_{\mathcal{C}}(x, y) = y$

$\mathcal{E} \vDash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$ $\qquad$ $\mathcal{E} \nvDash x + y \approx y + x$

Theorem (Birkhoff)

equational reasoning is sound and complete

$\forall$ *ES* $\mathcal{E}$: $\quad \approx_{\mathcal{E}} \; = \; =_{\mathcal{E}}$



Definition

ES $\mathcal{E}$ is consistent if $\exists$ terms s, t such that $s \not\approx_{\mathcal{E}} t$

Validity Problem

instance: ES $(\mathcal{F}, \mathcal{E})$ terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$
question: $s =_{\mathcal{E}} t$?

Theorem

validity problem is undecidable

Example (Combinatory Logic)

$$\mathsf{I} \cdot x \approx x$$
$$(\mathsf{K} \cdot x) \cdot y \approx x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$$

Outline

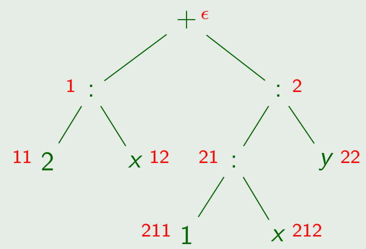
Definition (Positions)

- $\mathcal{P}os(\,\cdot\,)$

$$\mathcal{P}os(t) = \begin{cases} \{\epsilon\} & \text{if } t \in \mathcal{V} \\ \{\epsilon\} \cup \{ip \mid p \in \mathcal{P}os(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Example

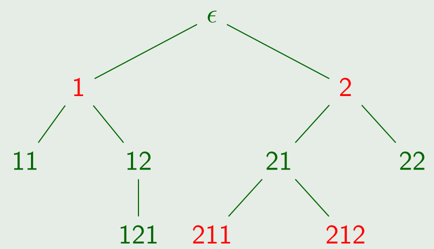
$(2 : x) + ((1 : x) : y)$



Definition

- $p < q$ if $\exists\, r \neq \epsilon$: $pr = q$ "p is strictly above q" "q is strictly below q"
- $p \leqslant q$ if $\exists\, r$: $pr = q$ "p is above q" "q is below q"
- $p \parallel q$ if $p \nleqslant q$ and $q \nleqslant p$ "p and q are parallel"

Example



- $2 < 211$
- $1 \parallel 2$
- $1 \parallel 212$

Definitions

- $t|_p$ subterm of t at position p

$$t|_p = \begin{cases} t & \text{if } p = \epsilon \\ t_i|_q & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

- $t(p)$ symbol in t at position p

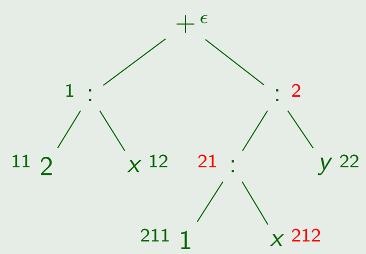
$$t(p) = \begin{cases} \mathrm{root}(t) & \text{if } p = \epsilon \\ t_i(q) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

- $t[s]_p$ replace subterm in t at position p by s

$$t[s]_p = \begin{cases} s & \text{if } p = \epsilon \\ f(t_1, \ldots, t_i[s]_q, \ldots, t_n) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } p = iq \end{cases}$$

Example

$t = (2 : x) + ((1 : x) : y)$



- $t|_{21} = 1 : x$
- $t(212) = x$
- $t[x + 3]_2 = (2 : x) + (x + 3)$

Definition

binary relation $\to_{\mathcal{E}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ for every ES $(\mathcal{F},\mathcal{E})$:

$$s \to_{\mathcal{E}} t \iff \begin{array}{l} \exists\ p \in \mathcal{P}\mathrm{os}(s) \\ \exists\ \ell \approx r \in \mathcal{E} \\ \exists \text{ substitution } \sigma \end{array} \quad \text{with} \quad \begin{array}{rcl} s|_p & = & \ell\sigma \\ t & = & s[r\sigma]_p \end{array} \quad \text{redex}$$

Example

ES $\mathcal{E} = \{0 + y \approx y,\ \mathsf{s}(x) + y \approx \mathsf{s}(x + y)\}$

	position	equation	substitution
$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0))$			
$\downarrow_{\mathcal{E}}$	1	$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$	$\{x \mapsto 0, y \mapsto \mathsf{s}(0)\}$
$\mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0)))$			
$\downarrow_{\mathcal{E}}$	11	$0 + y \approx y$	$\{y \mapsto \mathsf{s}(0)\}$
$\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$			

Lemma

$\to_{\mathcal{E}}$ is smallest relation that contains $\mathcal{E}$ and is closed under contexts and substitutions

Remark

with every ES $(\mathcal{F},\mathcal{E})$ we associate ARS $\langle \mathcal{T}(\mathcal{F},\mathcal{V}), \to_{\mathcal{E}} \rangle$

- notation ($\to_{\mathcal{E}}^*$, $\leftrightarrow_{\mathcal{E}}^*$, $\mathrm{NF}(\mathcal{E})$, ...)
- properties (SN, CR, ...)

are obtained for free

Theorem

$\forall$ *ES* $\mathcal{E}$ $\quad \leftrightarrow_{\mathcal{E}}^* \ = \ \approx_{\mathcal{E}} \ = \ =_{\mathcal{E}}$

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{\dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))}\,[a] \qquad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\,[a]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}\,[c]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}\,[t]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\,[c]$$

$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \leftrightarrow^*_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \to_{\mathcal{E}} \mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0))) \to_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$$

Definitions

- rewrite rule $(\ell \to r)$ is equation $\ell \approx r$ such that
  - $\ell \notin \mathcal{V}$
  - $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}(\ell)$
- term rewrite system (TRS) is pair $(\mathcal{F}, \mathcal{R})$ consisting of
  - $\mathcal{F}$ signature
  - $\mathcal{R}$ set of rewrite rules between terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$

Lemma

TRSs are ESs

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $s =_{\mathcal{E}} t$?

Theorem

validity problem is decidable for ES $\mathcal{E}$ if $\exists$ finite TRS $\mathcal{R}$ such that

1. *$\mathcal{R}$ is complete (confluent and terminating)*
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

Example (Group Theory)

signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

ES $\mathsf{e} \cdot x \approx x \quad x^- \cdot x \approx \mathsf{e} \quad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad \mathcal{E}$

theorems $\mathsf{e}^- \downarrow_{\mathcal{R}} \mathsf{e} \quad (x \cdot y)^- \downarrow_{\mathcal{R}} y^- \cdot z^-$

TRS $\mathcal{R}$

$$\begin{aligned} \mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\ x^- \cdot x &\to \mathsf{e} & x \cdot x^- &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathsf{e}^- &\to \mathsf{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

- $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} \implies \mathcal{E}$ has decidable validity problem
- how to compute $\mathcal{R}$? completion (lectures 5 & 6)
- how to prove termination of $\mathcal{R}$? LPO or KBO (lectures 4 & 7)

Example

TRS $\mathcal{R}$ modeling Sieve of Eratostheness for generating list of prime numbers

$$\begin{aligned} \mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y) \\ \mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x)) \\ \mathsf{head}(x : y) &\to x & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w) \\ \mathsf{tail}(x : y) &\to y & \mathsf{filter}(\mathsf{s}(x), y : z, w) &\to y : \mathsf{filter}(x, z, w) \end{aligned}$$

- $\mathcal{R}$ is confluent but not terminating

$$\mathsf{from}(0) \to 0 : \mathsf{from}(\mathsf{s}(0)) \to 0 : (\mathsf{s}(0) : \mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \to \cdots$$

- how to prove confluence of $\mathcal{R}$? orthogonality (lecture 8)
- $\exists$ non-terminating terms with (unique) normal form

$$\mathsf{head}(\mathsf{tail}(\mathsf{tail}(\mathsf{primes}))) \to^! \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))))$$

- how to compute normal forms in $\mathcal{R}$? strategy (lecture 9)

Example (Combinatory Logic)

$$\begin{aligned} \mathsf{I} \cdot x &\to x & \mathsf{I}\,x &\to x & \mathsf{I}\,x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x & (\mathsf{K}\,x)\,y &\to x & \mathsf{K}\,x\,y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\,x)\,y)\,z &\to (x\,z)\,(y\,z) & \mathsf{S}\,x\,y\,z &\to x\,z\,(y\,z) \end{aligned}$$

- applicative notation: surpress $\cdot$ and adopt left-association
- CL is confluent but not terminating

$$\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$$

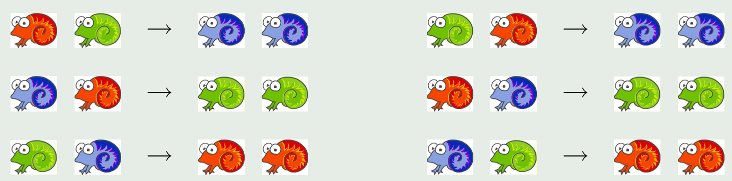
- CL is consistent

$$\mathsf{S} \not\leftrightarrow^* \mathsf{K}$$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is string rewrite system (SRS) if $\mathcal{F}$ consists of unary function symbols

Example



Theorem

term rewriting is Turing-complete hence all non-trivial questions are undecidable

Undecidable Problems

instance: (finite) TRS $\mathcal{R}$
question: is $\mathcal{R}$ terminating ?

instance: TRS $\mathcal{R}$
question: is $\mathcal{R}$ confluent ?

instance: TRS $\mathcal{R}$ term t
question: is t terminating ?

...

Theorem

- *confluence is decidable for terminating TRSs*
- *termination is undecidable for confluent TRSs*

Theorem

most problems for ground TRSs are decidable

Definitions

- rewrite rule $\ell \to r$ is right-ground if r is ground
- rewrite rule $\ell \to r$ is ground if ℓ and r are ground
- TRS is (right-)ground if all rewrite rules are (right-)ground

Theorem

validity problem is decidable for ground ESs *congruence closure*

Outline

- Equational Reasoning
- Algebras
- Term Rewriting
- Further Reading

- On the Uniform Halting Problem for Term Rewriting Systems
  Gérard Huet and Dallas Lankford
  technical report 283, INRIA, 1978
- Termination of Linear Rewriting Systems (Preliminary Version)
  Nachum Dershowitz
  Proc. 8th ICALP, pp. 448–458, 1981
- Fast Congruence Closure and Extensions
  Robert Nieuwenhuis and Albert Oliveras
  I&C 205(4), pp. 557–580, 2007