Introduction to Term Rewriting

lecture 4

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Introduction
- Well-Founded Monotone Algebras
- Polynomial Interpretations
- Lexicographic Path Order
- Further Reading

Definition

rewrite system is terminating if there are no infinite rewrite sequences

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Termination Methods 1967

Knuth-Bendix order

Definition

rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods 1975

Knuth-Bendix order, polynomial interpretations

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Termination Methods 1979

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order

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Termination Methods 1980s

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order

Definition

rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods 1990s

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling

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Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, . . .

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Termination Research

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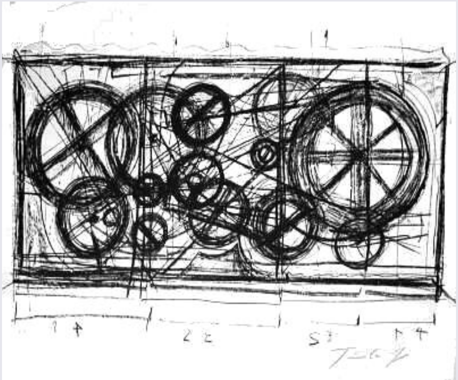

Termination Tools

AProVE, Cariboo, CiME, Jambox, Termptation, Matchbox, MuTerm, NTI, Torpa, TPA, $\text{T}_\text{T}\text{T}_2$, VMTL, . . .

Outline

Lemma

TRS $\mathcal{R}$ is terminating

$\iff$

$\exists$ *well-founded order* $>$ *on terms such that* $s > t$ *whenever* $s \to_{\mathcal{R}} t$

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Example

- TRS

$$0 + y \to y \qquad\qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

Lemma

TRS $\mathcal{R}$ is terminating

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$\exists$ *well-founded order $>$ on terms such that $s > t$ whenever $s \to_{\mathcal{R}} t$*

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- well-founded order $>$

$$s > t \quad \iff \quad \varphi(s) >_{\mathbb{N}} \varphi(t) \text{ with } \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = \mathsf{s}(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

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Remark

(very) inconvenient to check all rewrite steps

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- rewrite order is proper order $>$ on terms which is
  - closed under contexts $\quad s > t \implies C[s] > C[t]$
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Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >$ *for reduction order* $>$

Definitions

- well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of nonempty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_\mathcal{A}$ is strictly monotone in all coordinates:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) > f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$

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TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_\mathcal{A}$ for well-founded monotone algebra $(\mathcal{A}, >)$

Well-Founded Monotone Algebras

used in termination proofs/tools :

- polynomial interpretations over $\mathbb{N}$

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- $\dots$

Outline

Example

- TRS

$$0 + y \rightarrow y \qquad 0 \times y \rightarrow 0$$
$$\mathsf{s}(x) + y \rightarrow \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \rightarrow y + (x \times y)$$

Example

- TRS

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to y + (x \times y)$$

- interpretations in $\mathbb{N}$

$$0_{\mathcal{A}} = 1 \qquad\qquad +_{\mathcal{A}}(x, y) = 2x + y$$
$$\mathsf{s}_{\mathcal{A}}(x) = x + 1 \qquad\qquad \times_{\mathcal{A}}(x, y) = 2xy + x + y + 1$$

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- constraints $\forall x, y \in \mathbb{N}$

$$y + 2 > y \qquad 3y + 2 > 1$$

$$2x + y + 2 > 2x + y + 1 \qquad 2xy + x + 3y + 2 > 2xy + x + 3y + 1$$

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$$2 > 0 \qquad\qquad 3y + 1 > 0$$
$$1 > 0 \qquad\qquad 1 > 0$$

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- $\mathsf{s}(0) \times \mathsf{s}(\mathsf{s}(0)) \;\to\; \mathsf{s}(\mathsf{s}(0)) + (0 \times \mathsf{s}(\mathsf{s}(0))) \;\to\; \mathsf{s}(\mathsf{s}(0)) + 0 \;\to\; \mathsf{s}(\mathsf{s}(0) + 0)$

  $\to\; \mathsf{s}(\mathsf{s}(0 + 0)) \;\to\; \mathsf{s}(\mathsf{s}(0))$

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  $18 > 17 > 7 > 6$
  $\to \mathsf{s}(\mathsf{s}(0 + 0)) \to \mathsf{s}(\mathsf{s}(0))$
  $> 5 > 3$

Example

- TRS

$$
\begin{aligned}
\partial(x + y) &\rightarrow \partial(x) + \partial(y) & \partial(\alpha) &= 1 \\
\partial(x - y) &\rightarrow \partial(x) - \partial(y) & \partial(\beta) &= 0 \\
\partial(x \times y) &\rightarrow (\partial(x) \times y) + (x \times \partial(y)) \\
\partial(x \div y) &\rightarrow ((\partial(x) \times y) - (x \times \partial(y))) \div (y \times y)
\end{aligned}
$$

Example

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$$\partial(x+y) \to \partial(x)+\partial(y) \qquad \partial(\alpha) = 1$$
$$\partial(x-y) \to \partial(x)-\partial(y) \qquad \partial(\beta) = 0$$
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$$\alpha_{\mathcal{A}} = \beta_{\mathcal{A}} = 0_{\mathcal{A}} = 1_{\mathcal{A}} = 1$$
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$$\partial_{\mathcal{A}}(x) = x^2+6x+6$$

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- constraints $\quad \forall x, y \in \mathbb{N}$

$$x^2+y^2+2xy+12x+12y+33 > x^2+y^2+6x+6y+15 \qquad 13 > 1$$
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$$x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+7y+21$$
$$x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+9y+27$$

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- constraints $\forall x,y \in \mathbb{N}$

$$2xy+6x+6y+18 > 0 \qquad 13 > 1$$
$$2xy+6x+6y+18 > 0 \qquad 13 > 1$$
$$2xy+5x+5y+12 > 0$$
$$2xy+5x+3y+6 > 0$$

Definition

TRS $\mathcal{R}$ is polynomially terminating (over $\mathbb{N}$) if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of $\mathcal{A}$ is $\mathbb{N}$
- $>$ is standard order on $\mathbb{N}$
- $f_{\mathcal{A}} \in \mathbb{Z}[x_1, \ldots, x_n]$ for every n-ary f

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Lemma

$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$

$\iff$

$\mathcal{R}$ is polynomially terminating over $\{n \in \mathbb{N} \mid n \geqslant N\}$ for some $N \geqslant 0$

Questions

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following problem is undecidable:

instance: *polynomial* $P \in \mathbb{Z}[x_1, \ldots, x_n]$

question: $\forall x_1, \ldots, x_n \in \mathbb{N}\colon P(x_1, \ldots, x_n) > 0$?

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all coefficients are non-negative and constant is positive

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Proof

reduction from Hilbert's 10th Problem

for arbitrary polynomial $Q \in \mathbb{Z}[x_1, \ldots, x_n]$

$\exists x_1, \ldots, x_n \in \mathbb{Z}\colon Q(x_1, \ldots, x_n) = 0$

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$$\iff \quad \neg \forall x_1, \ldots, x_n \in \mathbb{Z}\colon Q(x_1, \ldots, x_n)^2 > 0$$

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$\iff \quad \neg\forall x_1, \dots, x_n \in \mathbb{Z}\colon Q(x_1, \dots, x_n)^2 > 0$

$\iff \quad \exists a_1, \dots, a_n \in \{-1, 1\}\ \neg\forall x_1, \dots, x_n \in \mathbb{N}\colon Q(a_1 x_1, \dots, a_n x_n)^2 > 0$

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$\iff \quad \neg\forall x_1, \ldots, x_n \in \mathbb{Z}\colon Q(x_1, \ldots, x_n)^2 > 0$

$\iff \quad \exists a_1, \ldots, a_n \in \{-1, 1\}\ \neg\forall x_1, \ldots, x_n \in \mathbb{N}\colon Q(a_1 x_1, \ldots, a_n x_n)^2 > 0$

$\in \mathbb{Z}[x_1, \ldots, x_n]$

Questions

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(e) translate resulting diophantine constraints to SAT or SMT problem

Example

- rewrite system

$$0 + y \to y$$
$$s(x) + y \to s(x + y)$$

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$$\begin{aligned} 0 + y &\rightarrow y \\ \mathsf{s}(x) + y &\rightarrow \mathsf{s}(x + y) \end{aligned}$$

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$$(e - 1)y + da + f > 0$$
$$(e - be)y + dc + f - bf - c > 0$$
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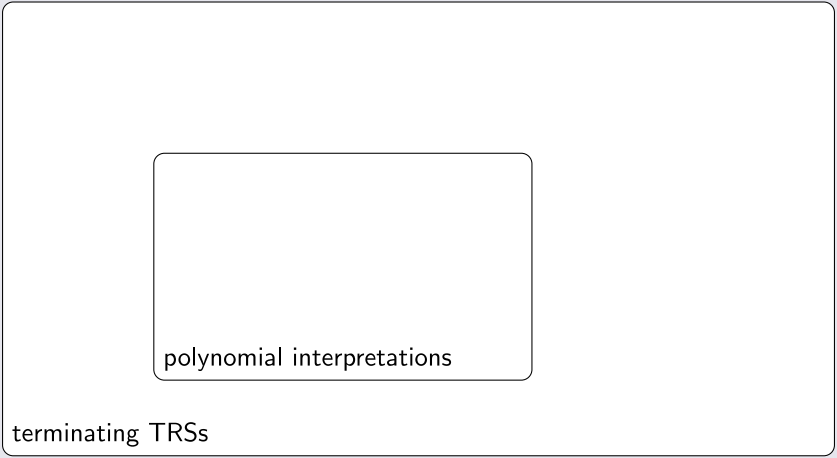

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Theorem

$>_{lpo}$ is reduction order if precedence $>$ is well-founded

Examples

TRS | precedence

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s(x) + y &\to s(x + y) \\
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$$e \cdot x \to x \qquad x \cdot e \to x$$
$$x^- \cdot x \to e \qquad x \cdot x^- \to e$$
$$(x \cdot y) \cdot z \to x \cdot (y \cdot z) \qquad x^{--} \to x$$
$$e^- \to e \qquad (x \cdot y)^- \to y^- \cdot x^-$$
$$x^- \cdot (x \cdot y) \to y \qquad x \cdot (x^- \cdot y) \to y$$

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     question: $\exists$ precedence $>$ such that $s >_{lpo} t$?

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LPO and polynomial interpretations are incomparable

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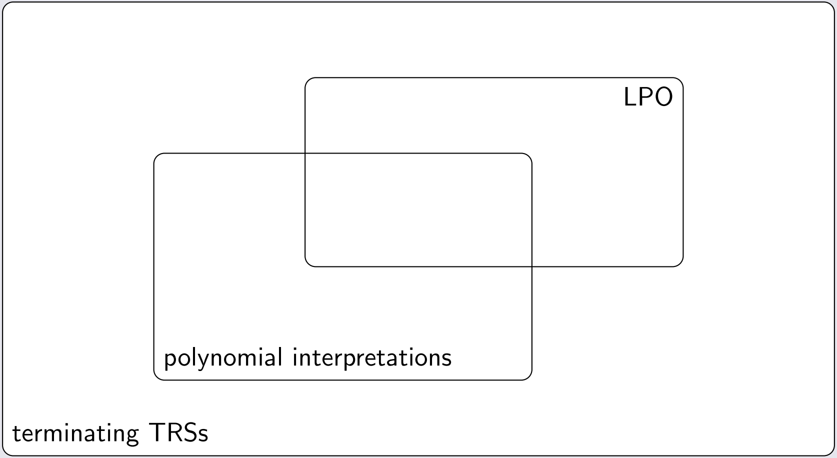

Outline

- Introduction
- Well-Founded Monotone Algebras
- Polynomial Interpretations
- Lexicographic Path Order
- **Further Reading**

- Termination of Term Rewriting: Interpretation and Type Elimination
  Hans Zantema
  JSC 17(1), pp. 23 – 50, 1994
- Mechanically Proving Termination Using Polynomial Interpretations
  Evelyne Contejean, Claude Marché, Ana Paula Tomás, and Xavier Urbain
  JAR 34(4), pp. 325 – 363, 2006
- Orderings for Term-Rewriting Systems
  Nachum Dershowitz
  TCS 17(3), pp. 279 – 301, 1982