



# Introduction to Term Rewriting

## lecture 4

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## Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

## Monday

termination, completion

## Tuesday

completion, termination

## Wednesday

confluence, modularity, strategies

## Thursday

exam, advanced topics

# Outline

- Introduction
- Well-Founded Monotone Algebras
- Polynomial Interpretations
- Lexicographic Path Order
- Further Reading

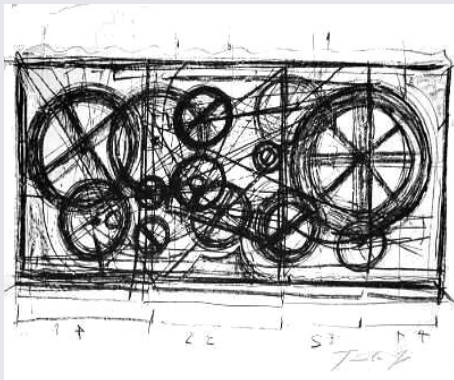
## Definition

rewrite system is **terminating** if there are no infinite rewrite sequences

## Termination Methods

**Knuth-Bendix order**, **polynomial interpretations**, multiset order, simple path order, **lexicographic path order**, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, **well-founded monotone algebras**, general path order, semantic labeling, dummy elimination, **dependency pairs**, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, ...

## Termination Research



## Termination Tools

AProVE, Cariboo, CiME, Jambox, Termptation, Matchbox, MuTerm, NTI, Torpa, TPA,  $T_1T_2$ , VMTL, ...

# Outline

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- **Well-Founded Monotone Algebras**
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## Lemma

TRS  $\mathcal{R}$  is terminating

$\iff$

$\exists$  well-founded order  $>$  on terms such that  $s > t$  whenever  $s \rightarrow_{\mathcal{R}} t$

## Example

- TRS

$$0 + y \rightarrow y$$

$$s(x) + y \rightarrow s(x + y)$$

- well-founded order  $>$

$$s > t \iff \varphi(s) >_{\mathbb{N}} \varphi(t) \text{ with } \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = s(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

## Remark

(very) inconvenient to check all rewrite *steps*

## Definitions

- **rewrite order** is proper order  $>$  on terms which is
  - closed under contexts  $s > t \implies C[s] > C[t]$
  - closed under substitutions  $s > t \implies s\sigma > t\sigma$
- TRS  $\mathcal{R}$  and rewrite order  $>$  are **compatible** if  $\ell > r$  for all rules  $\ell \rightarrow r$  in  $\mathcal{R}$
- **reduction order** is well-founded rewrite order

## Notation

$\mathcal{R} \subseteq >$  if  $\mathcal{R}$  and  $>$  are compatible

## Theorem

*TRS  $\mathcal{R}$  is terminating*  $\iff \mathcal{R} \subseteq >$  for reduction order  $>$



## Definitions

- **well-founded monotone  $\mathcal{F}$ -algebra**  $(\mathcal{A}, >)$  consists of nonempty algebra  $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$  together with well-founded order  $>$  on  $A$  such that every  $f_{\mathcal{A}}$  is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) > f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

for all  $a_1, \dots, a_n, b \in A$  and  $i \in \{1, \dots, n\}$  with  $a_i > b$

- relation  $>_{\mathcal{A}}$  on terms:  $s >_{\mathcal{A}} t$  if  $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$  for all assignments  $\alpha$

## Lemma

$>_{\mathcal{A}}$  is reduction order for every well-founded monotone algebra  $(\mathcal{A}, >)$

## Theorem

TRS  $\mathcal{R}$  is terminating  $\iff \mathcal{R} \subseteq >_{\mathcal{A}}$  for well-founded monotone algebra  $(\mathcal{A}, >)$

## Well-Founded Monotone Algebras

used in termination proofs/tools:

- polynomial interpretations over  $\mathbb{N}$
- polynomial interpretations over  $\mathbb{Q}$  and  $\mathbb{R}$
- matrix interpretations over  $\mathbb{N}$
- matrix interpretations over  $\mathbb{N} \cup \{-\infty\}$
- ...

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## Example

- TRS

$$\begin{aligned} 0 + y &\rightarrow y \\ s(x) + y &\rightarrow s(x + y) \end{aligned}$$

$$\begin{aligned} 0 \times y &\rightarrow 0 \\ s(x) \times y &\rightarrow y + (x \times y) \end{aligned}$$

- interpretations in  $\mathbb{N}$

$$\begin{aligned} 0_{\mathcal{A}} &= 1 & +_{\mathcal{A}}(x, y) &= 2x + y \\ s_{\mathcal{A}}(x) &= x + 1 & \times_{\mathcal{A}}(x, y) &= 2xy + x + y + 1 \end{aligned}$$

- constraints  $\forall x, y \in \mathbb{N}$

$$\begin{aligned} 2 &> 0 & 3y + 1 &> 0 \\ 1 &> 0 & 1 &> 0 \end{aligned}$$

- $s(0) \times s(s(0)) \rightarrow s(s(0)) + (0 \times s(s(0))) \rightarrow s(s(0)) + 0 \rightarrow s(s(0) + 0)$   
 $18 > 17 > 7 > 6$   
 $\rightarrow s(s(0 + 0)) \rightarrow s(s(0))$   
 $> 5 > 3$

## Example

- TRS

$$\partial(x + y) \rightarrow \partial(x) + \partial(y) \qquad \partial(\alpha) = 1$$

$$\partial(x - y) \rightarrow \partial(x) - \partial(y) \qquad \partial(\beta) = 0$$

$$\partial(x \times y) \rightarrow (\partial(x) \times y) + (x \times \partial(y))$$

$$\partial(x \div y) \rightarrow ((\partial(x) \times y) - (x \times \partial(y))) \div (y \times y)$$

- interpretations in  $\mathbb{N}$

$$\alpha_{\mathcal{A}} = \beta_{\mathcal{A}} = 0_{\mathcal{A}} = 1_{\mathcal{A}} = 1$$

$$+_{\mathcal{A}}(x, y) = -_{\mathcal{A}}(x, y) = \times_{\mathcal{A}}(x, y) = \div_{\mathcal{A}}(x, y) = x + y + 3$$

$$\partial_{\mathcal{A}}(x) = x^2 + 6x + 6$$

- constraints  $\forall x, y \in \mathbb{N}$

$$x^2 + y^2 + 2xy + 12x + 12y + 33 > x^2 + y^2 + 6x + 6y + 15 \qquad 13 > 1$$

$$x^2 + y^2 + 2xy + 12x + 12y + 33 > x^2 + y^2 + 6x + 6y + 15 \qquad 13 > 1$$

$$x^2 + y^2 + 2xy + 12x + 12y + 33 > x^2 + y^2 + 7x + 7y + 21$$

$$x^2 + y^2 + 2xy + 12x + 12y + 33 > x^2 + y^2 + 7x + 9y + 27$$

## Definition

TRS  $\mathcal{R}$  is **polynomially terminating (over  $\mathbb{N}$ )** if  $\mathcal{R} \subseteq >_{\mathcal{A}}$  for well-founded **monotone** algebra  $(\mathcal{A}, >)$  such that

- carrier of  $\mathcal{A}$  is  $\mathbb{N}$
  - $>$  is standard order on  $\mathbb{N}$
  - $f_{\mathcal{A}} \in \mathbb{Z}[x_1, \dots, x_n]$  for every  $n$ -ary  $f$
- polynomials with coefficients in  $\mathbb{Z}$   
and indeterminates  $x_1, \dots, x_n$

## Lemma

$\mathcal{R}$  is polynomially terminating over  $\mathbb{N}$

$\iff$

$\mathcal{R}$  is polynomially terminating over  $\{n \in \mathbb{N} \mid n \geq N\}$  for some  $N \geq 0$

## Questions

- 1 how to find suitable polynomials ?
- 2 how to show that  $P > 0$  for polynomial  $P \in \mathbb{Z}[x_1, \dots, x_n]$  ?

## Theorem

following problem is undecidable:

instance: polynomial  $P \in \mathbb{Z}[x_1, \dots, x_n]$

question:  $\forall x_1, \dots, x_n \in \mathbb{N}: P(x_1, \dots, x_n) > 0$  ?

## Sufficient Condition

all coefficients are non-negative and constant is positive (absolute positiveness)

## Theorem

following problem is undecidable:

instance: polynomial  $P \in \mathbb{Z}[x_1, \dots, x_n]$

question:  $\forall x_1, \dots, x_n \in \mathbb{N}: P(x_1, \dots, x_n) > 0$  ?

## Proof

reduction from **Hilbert's 10th Problem**

for arbitrary polynomial  $Q \in \mathbb{Z}[x_1, \dots, x_n]$

$\exists x_1, \dots, x_n \in \mathbb{Z}: Q(x_1, \dots, x_n) = 0$

$\iff \neg \forall x_1, \dots, x_n \in \mathbb{Z}: Q(x_1, \dots, x_n) \neq 0$

$\iff \neg \forall x_1, \dots, x_n \in \mathbb{Z}: Q(x_1, \dots, x_n)^2 > 0$

$\iff \exists a_1, \dots, a_n \in \{-1, 1\} \neg \forall x_1, \dots, x_n \in \mathbb{N}: Q(a_1x_1, \dots, a_nx_n)^2 > 0$

$\in \mathbb{Z}[x_1, \dots, x_n]$



## Questions

- 1 how to find suitable polynomials ?
- 2 how to show that  $P > 0$  for polynomial  $P \in \mathbb{Z}[x_1, \dots, x_n]$  ?

## Modern Approach

- (a) choose **abstract** polynomial interpretations (linear, quadratic, ...)
- (b) transform rewrite rules into polynomial ordering constraints
- (c) add monotonicity and well-definedness constraints
- (d) eliminate universally quantified variables using absolute positiveness
- (e) translate resulting diophantine constraints to SAT or SMT problem

## Example

- rewrite system

$$\begin{aligned}0 + y &\rightarrow y \\ s(x) + y &\rightarrow s(x + y)\end{aligned}$$

- interpretations

$$\begin{aligned}0_{\mathcal{A}} &= 0 \\ s_{\mathcal{A}}(x) &= x + 1 \\ +_{\mathcal{A}}(x, y) &= 2x + y + 1\end{aligned}$$

- diophantine constraints  $\forall x, y \in \mathbb{N}$

$$\begin{aligned}e - 1 &\geq 0 & da + f &> 0 \\ e - be &\geq 0 & dc + f - bf - c &> 0 \\ a &\geq 0 & b &\geq 1 & c &\geq 0 & d &\geq 1 & e &\geq 1 & f &\geq 0\end{aligned}$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

## Remark

*numerous terminating TRSs are not polynomially terminating*

polynomial interpretations

terminating TRSs

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## Definitions

- **precedence** is proper order  $>$  on  $\mathcal{F}$
- binary relation  $>_{lpo}$  on terms over  $\mathcal{F}$ :  
 $s >_{lpo} t$  if  $s = f(s_1, \dots, s_n)$  and either

- 1  $t = f(t_1, \dots, t_n)$  and  $\exists i$

$$\forall j < i \quad s_j = t_j \quad s_i >_{lpo} t_i \quad \forall j > i \quad s >_{lpo} t_j$$

- 2  $t = g(t_1, \dots, t_m)$  and  $f > g$  and  $\forall j \quad s >_{lpo} t_j$

- 3  $\exists i \quad s_i >_{lpo} t$  or  $s_i = t$

## Theorem

$>_{lpo}$  is **reduction order** if precedence  $>$  is well-founded

## Examples

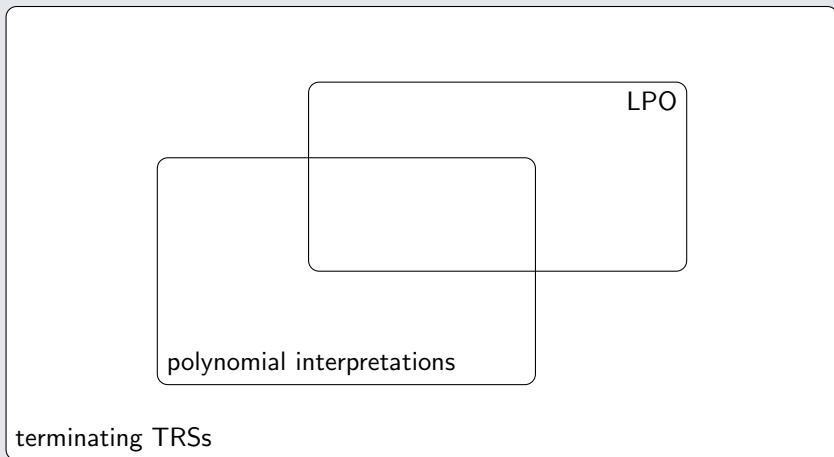
TRS	precedence
$0 + y \rightarrow y$ $s(x) + y \rightarrow s(x + y)$ $0 \times y \rightarrow 0$ $s(x) \times y \rightarrow (x \times y) + y$	$\times > + > s$
$ack(0, 0) \rightarrow 0$ $ack(0, s(y)) \rightarrow s(s(ack(0, y)))$ $ack(s(x), 0) \rightarrow s(0)$ $ack(s(x), s(y)) \rightarrow ack(x, ack(s(x), y))$	$ack > s$
$e \cdot x \rightarrow x$ $x^- \cdot x \rightarrow e$ $(x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z)$ $e^- \rightarrow e$ $x^- \cdot (x \cdot y) \rightarrow y$	$x \cdot e \rightarrow x$ $x \cdot x^- \rightarrow e$ $x^{--} \rightarrow x$ $(x \cdot y)^- \rightarrow y^- \cdot x^-$ $x \cdot (x^- \cdot y) \rightarrow y$
	$- > \cdot > e$

## Theorem

- if  $> \subseteq \sqsupset$  then  $>_{lpo} \subseteq \sqsupset_{lpo}$  (*incrementality*)
- if  $>$  is total then  $>_{lpo}$  is *total on ground terms*
- following two problems are *decidable*:
  - 1 instance: terms  $s, t$  precedence  $>$   
question:  $s >_{lpo} t ?$
  - 2 instance: terms  $s, t$   
question:  $\exists$  precedence  $>$  such that  $s >_{lpo} t ?$

## Remark

*LPO and polynomial interpretations are incomparable*





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## Termination of Term Rewriting: Interpretation and Type Elimination

Hans Zantema

JSC 17(1), pp. 23 – 50, 1994



## Mechanically Proving Termination Using Polynomial Interpretations

Evelyne Contejean, Claude Marché, Ana Paula Tomás, and Xavier Urbain

JAR 34(4), pp. 325 – 363, 2006



## Orderings for Term-Rewriting Systems

Nachum Dershowitz

TCS 17(3), pp. 279 – 301, 1982