Introduction to Term Rewriting

lecture 4

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Introduction
- Well-Founded Monotone Algebras
- Polynomial Interpretations
- Lexicographic Path Order
- Further Reading

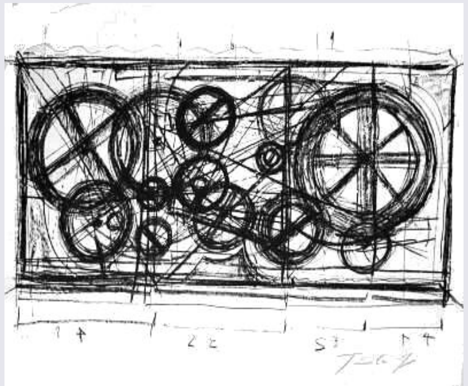
Definition

rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, . . .

Termination Research



Termination Tools

AProVE, Cariboo, CiME, Jambox, Termptation, Matchbox, MuTerm, NTI, Torpa, TPA, $\mathsf{T_TT_2}$, VMTL, . . .

Outline

Lemma

TRS $\mathcal{R}$ is terminating

$\iff$

$\exists$ *well-founded order $>$ on terms such that $s > t$ whenever $s \to_{\mathcal{R}} t$*

Example

- TRS

$$0 + y \to y \qquad\qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- well-founded order $>$

$$s > t \iff \varphi(s) >_{\mathbb{N}} \varphi(t) \text{ with } \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = \mathsf{s}(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

Remark

(very) inconvenient to check all rewrite steps

Definitions

- rewrite order is proper order $>$ on terms which is
  - closed under contexts $\quad s > t \implies C[s] > C[t]$
  - closed under substitutions $\quad s > t \implies s\sigma > t\sigma$
- TRS $\mathcal{R}$ and rewrite order $>$ are compatible if $\ell > r$ for all rules $\ell \to r$ in $\mathcal{R}$
- reduction order is well-founded rewrite order

Notation

$\mathcal{R} \subseteq >$ if $\mathcal{R}$ and $>$ are compatible

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ *$\mathcal{R} \subseteq >$ for reduction order $>$*

Definitions

- well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of nonempty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) > f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$
- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemma

$>_{\mathcal{A}}$ is reduction order for every well-founded monotone algebra $(\mathcal{A}, >)$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $\mathcal{R} \subseteq >_{\mathcal{A}}$ for well-founded monotone algebra $(\mathcal{A}, >)$

Well-Founded Monotone Algebras

used in termination proofs/tools :

- polynomial interpretations over $\mathbb{N}$
- polynomial interpretations over $\mathbb{Q}$ and $\mathbb{R}$
- matrix interpretations over $\mathbb{N}$
- matrix interpretations over $\mathbb{N} \cup \{-\infty\}$
- $\ldots$

Outline

Example

- TRS

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$

$$s(x) + y \to s(x + y) \qquad\qquad s(x) \times y \to y + (x \times y)$$

- interpretations in $\mathbb{N}$

$$0_{\mathcal{A}} = 1 \qquad\qquad +_{\mathcal{A}}(x, y) = 2x + y$$

$$s_{\mathcal{A}}(x) = x + 1 \qquad\qquad \times_{\mathcal{A}}(x, y) = 2xy + x + y + 1$$

- constraints $\quad \forall x, y \in \mathbb{N}$

$$2 > 0 \qquad\qquad 3y + 1 > 0$$

$$1 > 0 \qquad\qquad 1 > 0$$

- $\underset{18}{s(0) \times s(s(0))} \underset{>}{\to} \underset{17}{s(s(0)) + (0 \times s(s(0)))} \underset{>}{\to} \underset{7}{s(s(0)) + 0} \underset{>}{\to} \underset{6}{s(s(0) + 0)}$

  $\underset{>}{\to} \underset{5}{s(s(0 + 0))} \underset{>}{\to} \underset{3}{s(s(0))}$

Example

- TRS

$$\begin{aligned} \partial(x+y) &\to \partial(x)+\partial(y) & \partial(\alpha) &= 1 \\ \partial(x-y) &\to \partial(x)-\partial(y) & \partial(\beta) &= 0 \\ \partial(x\times y) &\to (\partial(x)\times y)+(x\times\partial(y)) \\ \partial(x\div y) &\to ((\partial(x)\times y)-(x\times\partial(y)))\div(y\times y) \end{aligned}$$

- interpretations in $\mathbb{N}$

$$\begin{aligned} \alpha_\mathcal{A} &= \beta_\mathcal{A} = 0_\mathcal{A} = 1_\mathcal{A} = 1 \\ +_\mathcal{A}(x,y) &= -_\mathcal{A}(x,y) = \times_\mathcal{A}(x,y) = \div_\mathcal{A}(x,y) = x+y+3 \\ \partial_\mathcal{A}(x) &= x^2+6x+6 \end{aligned}$$

- constraints $\quad \forall x,y \in \mathbb{N}$

$$\begin{aligned} x^2+y^2+2xy+12x+12y+33 &> x^2+y^2+6x+6y+15 & 13 &> 1 \\ x^2+y^2+2xy+12x+12y+33 &> x^2+y^2+6x+6y+15 & 13 &> 1 \\ x^2+y^2+2xy+12x+12y+33 &> x^2+y^2+7x+7y+21 \\ x^2+y^2+2xy+12x+12y+33 &> x^2+y^2+7x+9y+27 \end{aligned}$$

Definition

TRS $\mathcal{R}$ is polynomially terminating (over $\mathbb{N}$) if $\mathcal{R} \subseteq >_\mathcal{A}$ for well-founded monotone algebra $(\mathcal{A}, >)$ such that

- carrier of $\mathcal{A}$ is $\mathbb{N}$
- $>$ is standard order on $\mathbb{N}$
- $f_\mathcal{A} \in \mathbb{Z}[x_1, \ldots, x_n]$ for every n-ary f

polynomials with coefficients in $\mathbb{Z}$ and indeterminates $x_1, \ldots, x_n$

Lemma

$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$

$\Longleftrightarrow$

$\mathcal{R}$ is polynomially terminating over $\{n \in \mathbb{N} \mid n \geqslant N\}$ for some $N \geqslant 0$

Questions

1. how to find suitable polynomials ?
2. how to show that $P > 0$ for polynomial $P \in \mathbb{Z}[x_1,\dots,x_n]$?

Theorem

following problem is undecidable:

instance: polynomial $P \in \mathbb{Z}[x_1,\dots,x_n]$

question: $\forall x_1,\dots,x_n \in \mathbb{N}\colon P(x_1,\dots,x_n) > 0$?

Sufficient Condition

all coefficients are non-negative and constant is positive (absolute positiveness)

Theorem

following problem is undecidable:

instance: polynomial $P \in \mathbb{Z}[x_1,\dots,x_n]$

question: $\forall x_1,\dots,x_n \in \mathbb{N}\colon P(x_1,\dots,x_n) > 0$?

Proof

reduction from Hilbert's 10th Problem

for arbitrary polynomial $Q \in \mathbb{Z}[x_1,\dots,x_n]$

$\exists x_1,\dots,x_n \in \mathbb{Z}\colon Q(x_1,\dots,x_n) = 0$

$\iff \neg\forall x_1,\dots,x_n \in \mathbb{Z}\colon Q(x_1,\dots,x_n) \neq 0$

$\iff \neg\forall x_1,\dots,x_n \in \mathbb{Z}\colon Q(x_1,\dots,x_n)^2 > 0$

$\iff \exists a_1,\dots,a_n \in \{-1,1\}\ \neg\forall x_1,\dots,x_n \in \mathbb{N}\colon Q(a_1x_1,\dots,a_nx_n)^2 > 0$

$\in \mathbb{Z}[x_1,\dots,x_n]$

Questions

1. how to find suitable polynomials ?
2. how to show that $P > 0$ for polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$?

Modern Approach

(a) choose abstract polynomial interpretations (linear, quadratic, ...)

(b) transform rewrite rules into polynomial ordering constraints

(c) add monotonicity and well-definedness constraints

(d) eliminate universally quantified variables using absolute positiveness

(e) translate resulting diophantine constraints to SAT or SMT problem

Example

- rewrite system

$$0 + y \to y$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- interpretations

$$0_{\mathcal{A}} = 0$$
$$\mathsf{s}_{\mathcal{A}}(x) = x + 1$$
$$+_{\mathcal{A}}(x, y) = 2x + y + 1$$

- diophantine constraints $\forall x, y \in \mathbb{N}$

$$e - 1 \geqslant 0 \quad da + f > 0$$
$$e - be \geqslant 0 \quad dc + f - bf - c > 0$$
$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Remark

numerous terminating TRSs are not polynomially terminating

Outline

Definitions

- precedence is proper order $>$ on $\mathcal{F}$
- binary relation $>_{\text{lpo}}$ on terms over $\mathcal{F}$:
  $s >_{\text{lpo}} t$ if $s = f(s_1, \ldots, s_n)$ and either
  1. $t = f(t_1, \ldots, t_n)$ and $\exists\, i$

     $$\forall\, j < i \;\; s_j = t_j \qquad s_i >_{\text{lpo}} t_i \qquad \forall\, j > i \;\; s >_{\text{lpo}} t_j$$
  2. $t = g(t_1, \ldots, t_m)$ and $f > g$ and $\forall\, j \;\; s >_{\text{lpo}} t_j$
  3. $\exists\, i \;\; s_i >_{\text{lpo}} t$ or $s_i = t$

Theorem

$>_{lpo}$ is reduction order if precedence $>$ is well-founded

Examples

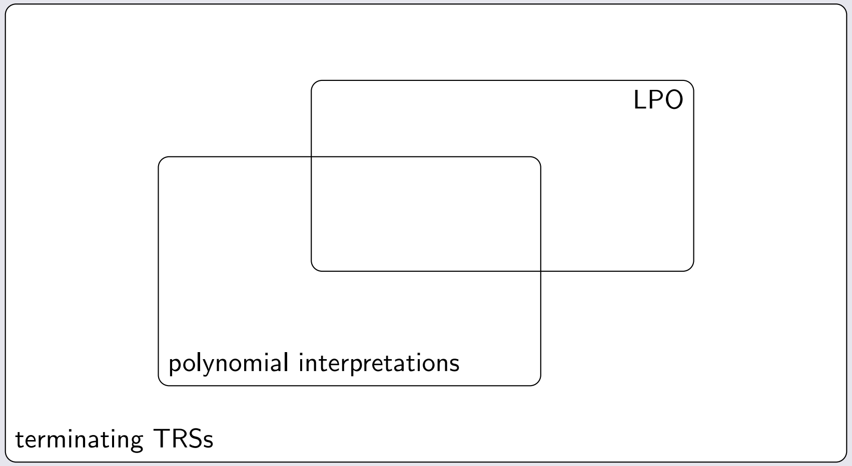
TRS		precedence
$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$ $0 \times y \to 0$ $\mathsf{s}(x) \times y \to (x \times y) + y$		$\times > + > \mathsf{s}$
$\mathsf{ack}(0,0) \to 0$ $\mathsf{ack}(0,\mathsf{s}(y)) \to \mathsf{s}(\mathsf{s}(\mathsf{ack}(0,y)))$ $\mathsf{ack}(\mathsf{s}(x),0) \to \mathsf{s}(0)$ $\mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)) \to \mathsf{ack}(x,\mathsf{ack}(\mathsf{s}(x),y))$		$\mathsf{ack} > \mathsf{s}$
$\mathsf{e} \cdot x \to x$ $x^- \cdot x \to \mathsf{e}$ $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ $\mathsf{e}^- \to \mathsf{e}$ $x^- \cdot (x \cdot y) \to y$	$x \cdot \mathsf{e} \to x$ $x \cdot x^- \to \mathsf{e}$ $x^{--} \to x$ $(x \cdot y)^- \to y^- \cdot x^-$ $x \cdot (x^- \cdot y) \to y$	$^- > \cdot > \mathsf{e}$

Theorem

- *if* $> \subseteq \sqsupset$ *then* $>_{lpo} \subseteq \sqsupset_{lpo}$ *(incrementality)*
- *if* $>$ *is total then* $>_{lpo}$ *is total on ground terms*
- *following two problems are decidable:*
  1. *instance:* terms s, t precedence $>$
     question: $s >_{lpo} t$?
  2. *instance:* terms s, t
     question: $\exists$ precedence $>$ such that $s >_{lpo} t$?

Remark

LPO and polynomial interpretations are incomparable



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