ISR 2010

Introduction to Term Rewriting

lecture 5

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Critical Pairs
- Unification
- Completion

Newman's Lemma

SN & WCR $\Rightarrow$ CR

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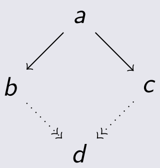
SN & WCR $\Rightarrow$ CR

Definition (WCR)

$\forall a, b, c$



$\exists d$

Newman's Lemma

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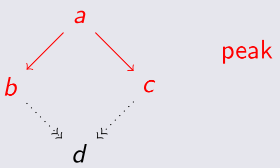

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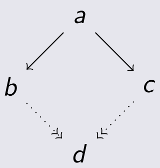

peak

$\exists d$

Question

how to prove WCR ?

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three peaks

$$\mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})})$$

1 ↙ ↘ 2

$\mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{b}))$ $\qquad$ $\mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{c})$

Example

$$\mathsf{f}(\mathsf{a},\mathsf{g}(x)) \to \mathsf{f}(x,x) \qquad\qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

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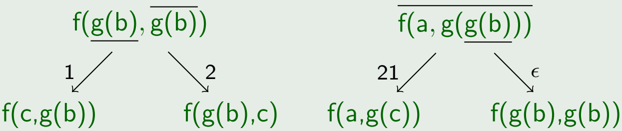

Example

$$f(a, g(x)) \to f(x, x) \qquad g(b) \to c$$

three peaks

$$f(c,g(b)) \xleftarrow{1} f(\underline{g(b)}, \overline{g(b)}) \xrightarrow{2} f(g(b),c)$$

$$f(a,g(c)) \xleftarrow{21} \overline{f(a, g(\underline{g(b)}))} \xrightarrow{\epsilon} f(g(b),g(b))$$

$$f(a,c) \xleftarrow{2} \overline{f(a, \underline{g(b)})} \xrightarrow{\epsilon} f(b,b)$$

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad\qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three peaks

$$\begin{array}{ccc} & \mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})}) & \\ {}^{1}\swarrow & & \searrow^{2} \\ \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{b})) & & \mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{c}) \end{array} \qquad \begin{array}{ccc} & \overline{\mathsf{f}(\mathsf{a}, \mathsf{g}(\underline{\mathsf{g}(\mathsf{b})}))} & \\ {}^{21}\swarrow & & \searrow^{\epsilon} \\ \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c})) & & \mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{g}(\mathsf{b})) \end{array} \qquad \begin{array}{ccc} & \overline{\mathsf{f}(\mathsf{a}, \underline{\mathsf{g}(\mathsf{b})})} & \\ {}^{2}\swarrow & & \searrow^{\epsilon} \\ \mathsf{f}(\mathsf{a},\mathsf{c}) & & \mathsf{f}(\mathsf{b},\mathsf{b}) \end{array}$$

parallel redexes

Example

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three peaks

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$$\mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{b})) \overset{1}{\leftarrow} \mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})}) \overset{2}{\to} \mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{c})$$

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parallel redexes

variable overlap

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$f(a,c) \leftarrow_{2} \overline{f(a, \underline{g(b)})} \to_{\epsilon} f(b,b)$

overlapping redexes

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parallel redexes	variable overlap	overlapping redexes
non-critical	non-critical	critical

Definitions

- overlap is triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are rewrite rules without common variables
  2. $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$
  3. ℓ_1 and $\ell_2|_p$ are unifiable
  4. if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants

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- $\ell_2\sigma[r_1\sigma]_p \leftarrow \ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma \to r_2\sigma$

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Critical Pair Lemma (Huet 1980)

TRS is locally confluent $\iff$ all critical pairs are convergent

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

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- $\langle\, (u \cdot v) \cdot w \to u \cdot (v \cdot w),\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

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critical pairs

- $u \cdot z \leftarrow\!\rtimes\!\to e \cdot (u \cdot z)$
- $e \cdot z \leftarrow\!\rtimes\!\to u^- \cdot (u \cdot z)$
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\!\rtimes\!\to (u \cdot v) \cdot (w \cdot z)$

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critical pairs

- $u \cdot z \leftarrow \rtimes \rightarrow e \cdot (u \cdot z)$ convergent
- $e \cdot z \leftarrow \rtimes \rightarrow u^- \cdot (u \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow \rtimes \rightarrow (u \cdot v) \cdot (w \cdot z)$ convergent

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critical pairs

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- $(u \cdot (v \cdot w)) \cdot z \leftarrow\rtimes\rightarrow (u \cdot v) \cdot (w \cdot z)$ convergent

Theorem (Knuth & Bendix 1970)

terminating TRS is confluent $\iff$ all critical pairs are convergent

Outline

- Critical Pairs
- **Unification**
- Completion

Definition

composition of substitutions σ and τ:

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Lemma

$(\rho\sigma)\tau = \rho(\sigma\tau)$ *for all substitutions* ρ, σ, τ

Definitions

- $\dot{\leq}$ subsumption

  $s \dot{\leq} t \iff \exists\sigma : s\sigma = t$

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  $s \leq t \iff \exists\sigma : s\sigma = t$ "s subsumes t" "t is instance of s"

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Example

$x + y \lesssim \mathsf{s}(y) + \mathsf{s}(0)$

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$x + y \lessdot \mathsf{s}(y) + \mathsf{s}(0)$ $\mathsf{s}(x) + y \not\lessdot x + \mathsf{s}(0)$

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Example

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Lemma

$\cdot\!\!>$ *is well-founded order on terms*

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Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \doteq \mathsf{s}(y) + \mathsf{s}(z + 0)$

Definitions

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Definition (Unification Problem)

instance: terms s, t

question: $\exists \underbrace{\text{substitution } \sigma}_{\text{unifier}}\colon\ s\sigma = t\sigma$?

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Definition

most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

d decomposition

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n),\ E_2}{E_1,\ s_1 \approx t_1,\ \ldots,\ s_n \approx t_n,\ E_2}$$

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t removal of trivial equations $(x \in \mathcal{V})$

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t removal of trivial equations $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx x,\ E_2}{E_1,\ E_2}$$

v variable elimination $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx t,\ E_2}{(E_1,\ E_2)\sigma} \quad \text{and} \quad \frac{E_1,\ t \approx x,\ E_2}{(E_1,\ E_2)\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

Example

$$x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)$$

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$$\mathsf{d} \Downarrow$$

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$$0 + \mathsf{s}(y) \approx 0 + \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$0 \approx 0,\ \mathsf{s}(y) \approx \mathsf{s}(z)$$

Example

$$x + (0 + s(y)) \approx s(z) + (0 + x)$$

$$\mathsf{d} \Downarrow$$

$$x \approx s(z),\ 0 + s(y) \approx 0 + x$$

$$\mathsf{v} \Downarrow \ x \mapsto s(z)$$

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$$\mathsf{d} \Downarrow$$

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$$\mathsf{d} \Downarrow$$

$$s(y) \approx s(z)$$

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$$\mathsf{d} \Downarrow$$

$$\mathsf{s}(y) \approx \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$y \approx z$$

Example

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$$\mathsf{d} \Downarrow$$

$$x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x$$

$$\mathsf{v} \Downarrow \quad x \mapsto \mathsf{s}(z)$$

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$$x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)$$

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$$x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x$$

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$$\square$$

Example

$$x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)$$

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$$x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x$$

$$\mathsf{v} \Downarrow \quad x \mapsto \mathsf{s}(z)$$

$$0 + \mathsf{s}(y) \approx 0 + \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$0 \approx 0,\ \mathsf{s}(y) \approx \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$\mathsf{s}(y) \approx \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$y \approx z$$

$$\mathsf{v} \Downarrow \quad y \mapsto z$$

$$\Box$$

mgu $\{x \mapsto \mathsf{s}(z), y \mapsto z\}$

Theorem

- *there are no infinite derivations*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots$$

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  - $\sigma_1\sigma_2\sigma_3 \cdots \sigma_n$ *is mgu of s and t*

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  - $E_n = \square$
  - $\sigma_1 \sigma_2 \sigma_3 \cdots \sigma_n$ *is mgu of s and t*

Optional Failure Rules

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx g(t_1,\ldots,t_m),\ E_2}{\bot} \qquad \frac{E_1,\ x \approx t,\ E_2}{\bot} \quad \frac{E_1,\ t \approx x,\ E_2}{\bot}$$

if $x \in \mathcal{V}\text{ar}(t)$

Outline

- Critical Pairs
- Unification
- Completion
  - Example
  - Procedure

Example

TRS $\mathcal{R}$

①	$x + 0 \rightarrow x$	②	$x - 0 \rightarrow x$
③	$x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

Example

TRS $\mathcal{R}$

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

- SN ?

Example

TRS $\mathcal{R}$

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$

Example

TRS $\mathcal{R}$

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
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- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ?

Example

TRS $\mathcal{R}$

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}}$: ⑥ $\swarrow$ $x + y$, ③ $\searrow$ $\mathsf{s}(x + \mathsf{p}(y))$

$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}}$: ⑥ $\swarrow$ $x - y$, ④ $\searrow$ $\mathsf{p}(x - \mathsf{p}(y))$

$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})}$: ⑥ $\swarrow$ $\mathsf{p}(x)$, ⑤ $\searrow$ $\mathsf{p}(x)$

$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})}$: ⑤ $\swarrow$ $\mathsf{s}(x)$, ⑥ $\searrow$ $\mathsf{s}(x)$

Example

TRS $\mathcal{R}$

①	$x + 0 \rightarrow x$	②	$x - 0 \rightarrow x$
③	$x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{⑥} x + y \qquad \xrightarrow{③} \mathsf{s}(x + \mathsf{p}(y))$$

$$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{⑥} x - y \qquad \xrightarrow{④} \mathsf{p}(x - \mathsf{p}(y))$$

$$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \quad \xrightarrow{⑥} \mathsf{p}(x) \quad = \quad \mathsf{p}(x) \xleftarrow{⑤}$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \quad \xrightarrow{⑤} \mathsf{s}(x) \quad = \quad \mathsf{s}(x) \xleftarrow{⑥}$$

Example

TRS $\mathcal{R}$

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$		

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}}: \quad x + y \;\overset{⑥}{\leftarrow}\; \cdot \;\overset{③}{\rightarrow}\; \mathsf{s}(x + \mathsf{p}(y)), \qquad x + y \;\overset{⑦}{\longleftarrow}\; \mathsf{s}(x + \mathsf{p}(y))$$

$$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}}: \quad x - y \;\overset{⑥}{\leftarrow}\; \cdot \;\overset{④}{\rightarrow}\; \mathsf{p}(x - \mathsf{p}(y))$$

$$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})}: \quad \mathsf{p}(x) \;\overset{⑥}{\leftarrow}\; \cdot \;\overset{⑤}{\rightarrow}\; \mathsf{p}(x), \qquad \mathsf{p}(x) = \mathsf{p}(x)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})}: \quad \mathsf{s}(x) \;\overset{⑤}{\leftarrow}\; \cdot \;\overset{⑥}{\rightarrow}\; \mathsf{s}(x), \qquad \mathsf{s}(x) = \mathsf{s}(x)$$

Example

TRS $\mathcal{R}$

①	$x + 0$	$\to$	x	②	$x - 0$	$\to$	x
③	$x + \mathsf{s}(y)$	$\to$	$\mathsf{s}(x + y)$	④	$x - \mathsf{s}(y)$	$\to$	$\mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x))$	$\to$	x	⑥	$\mathsf{s}(\mathsf{p}(x))$	$\to$	x
⑦	$\mathsf{s}(x + \mathsf{p}(y))$	$\to$	$x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y))$	$\to$	$x - y$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}}$: ⑥ $\swarrow$ $x + y$; ③ $\searrow$ $\mathsf{s}(x + \mathsf{p}(y))$; $x + y \xleftarrow{⑦} \mathsf{s}(x + \mathsf{p}(y))$

$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}}$: ⑥ $\swarrow$ $x - y$; ④ $\searrow$ $\mathsf{p}(x - \mathsf{p}(y))$; $x - y \xleftarrow{⑧} \mathsf{p}(x - \mathsf{p}(y))$

$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})}$: ⑥ $\swarrow$ $\mathsf{p}(x)$; ⑤ $\searrow$ $\mathsf{p}(x)$; $\mathsf{p}(x) = \mathsf{p}(x)$

$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})}$: ⑤ $\swarrow$ $\mathsf{s}(x)$; ⑥ $\searrow$ $\mathsf{s}(x)$; $\mathsf{s}(x) = \mathsf{s}(x)$

Example (cont'd)

- added rewrite rules

$$⑦ \quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad ⑧ \quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

preserve termination

Example (cont'd)

- added rewrite rules

$$⑦ \quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad ⑧ \quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

preserve termination and do not change $\overset{*}{\longleftrightarrow}$

Example (cont'd)

- added rewrite rules

$$⑦\quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad ⑧\quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

preserve termination and do not change $\overset{*}{\longleftrightarrow}$

- new critical pairs

$$\overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \qquad \mathsf{p}(x+y) \xleftarrow{⑦} \cdot \xrightarrow{⑤} x + \mathsf{p}(y)$$

$$\overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \qquad \mathsf{s}(x+y) \xleftarrow{⑤} \cdot \xrightarrow{⑦} x + \mathsf{s}(y)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \qquad \mathsf{s}(x-y) \xleftarrow{⑧} \cdot \xrightarrow{⑥} x - \mathsf{p}(y)$$

$$\overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \qquad \mathsf{p}(x-y) \xleftarrow{⑤} \cdot \xrightarrow{⑧} x - \mathsf{s}(y)$$

Example (cont'd)

- added rewrite rules

$$\textcircled{7} \quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad \textcircled{8} \quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

preserve termination and do not change $\overset{*}{\longleftrightarrow}$

- new critical pairs

$$\mathsf{p}(x + y) \xleftarrow{\textcircled{7}} \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \xrightarrow{\textcircled{5}} x + \mathsf{p}(y)$$

$$\mathsf{s}(x + y) \xleftarrow{\textcircled{5}} \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \xrightarrow{\textcircled{7}} x + \mathsf{s}(y), \qquad \mathsf{s}(x + y) \xleftarrow[\textcircled{2}]{} x + \mathsf{s}(y)$$

$$\mathsf{s}(x - y) \xleftarrow{\textcircled{8}} \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \xrightarrow{\textcircled{6}} x - \mathsf{p}(y)$$

$$\mathsf{p}(x - y) \xleftarrow{\textcircled{5}} \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \xrightarrow{\textcircled{8}} x - \mathsf{s}(y), \qquad \mathsf{p}(x - y) \xleftarrow[\textcircled{4}]{} x - \mathsf{s}(y)$$

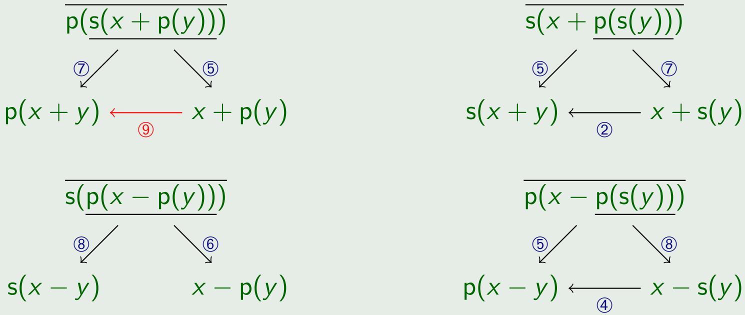
Example (cont'd)

- added rewrite rules

$$⑦ \quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad ⑧ \quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

  preserve termination and do not change $\overset{*}{\longleftrightarrow}$

- new critical pairs



Example (cont'd)

- added rewrite rules

$$⑦ \quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad ⑧ \quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

preserve termination and do not change $\overset{*}{\longleftrightarrow}$

- new critical pairs

$$\mathsf{p}(x + y) \xleftarrow{⑦} \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \xrightarrow{⑤} x + \mathsf{p}(y) \qquad \mathsf{p}(x + y) \xleftarrow[⑨]{} x + \mathsf{p}(y)$$

$$\mathsf{s}(x + y) \xleftarrow{⑤} \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \xrightarrow{⑦} x + \mathsf{s}(y) \qquad \mathsf{s}(x + y) \xleftarrow[②]{} x + \mathsf{s}(y)$$

$$\mathsf{s}(x - y) \xleftarrow{⑧} \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \xrightarrow{⑥} x - \mathsf{p}(y) \qquad \mathsf{s}(x - y) \xleftarrow[⑩]{} x - \mathsf{p}(y)$$

$$\mathsf{p}(x - y) \xleftarrow{⑤} \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \xrightarrow{⑧} x - \mathsf{s}(y) \qquad \mathsf{p}(x - y) \xleftarrow[④]{} x - \mathsf{s}(y)$$

Example (cont'd)

- added rewrite rules

$$⑨ \quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩ \quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

Example (cont'd)

- added rewrite rules

$$⑨ \quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩ \quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- new critical pairs

$$\overline{x + \underline{\mathsf{s}(y + \mathsf{p}(z))}}$$

$$x + (y + z) \;\overset{⑦}{\longleftarrow}\; \cdot \;\overset{②}{\longrightarrow}\; \mathsf{s}(x + (y + \mathsf{p}(z)))$$

$$\overline{x - \underline{\mathsf{s}(y + \mathsf{p}(z))}}$$

$$x - (y + z) \;\overset{⑦}{\longleftarrow}\; \cdot \;\overset{④}{\longrightarrow}\; \mathsf{p}(x - (y + \mathsf{p}(z)))$$

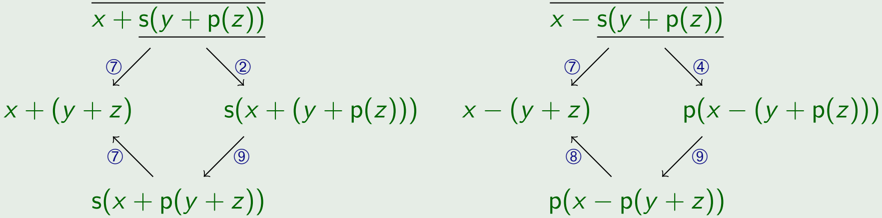
Example (cont'd)

- added rewrite rules

$$⑨ \quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩ \quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- new critical pairs



Example (cont'd)

- new critical pairs

$$\overline{\mathsf{s}(x + \underline{\mathsf{p}(y - \mathsf{p}(z))})}$$
$$\mathsf{s}(x + (y - z)) \xleftarrow{(8)} \cdot \xrightarrow{(7)} x + (y - \mathsf{p}(z))$$

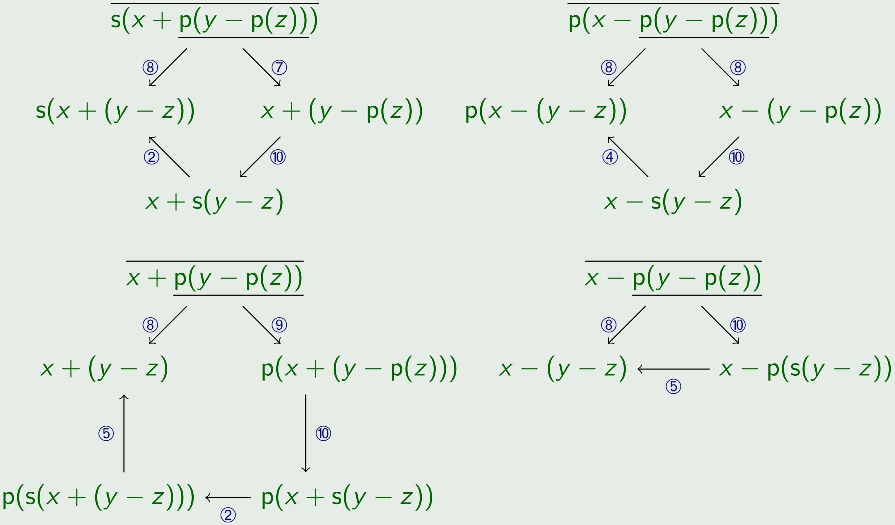
$$\overline{\mathsf{p}(x - \underline{\mathsf{p}(y - \mathsf{p}(z))})}$$
$$\mathsf{p}(x - (y - z)) \xleftarrow{(8)} \cdot \xrightarrow{(8)} x - (y - \mathsf{p}(z))$$

$$\overline{x + \underline{\mathsf{p}(y - \mathsf{p}(z))}}$$
$$x + (y - z) \xleftarrow{(8)} \cdot \xrightarrow{(9)} \mathsf{p}(x + (y - \mathsf{p}(z)))$$

$$\overline{x - \underline{\mathsf{p}(y - \mathsf{p}(z))}}$$
$$x - (y - z) \xleftarrow{(8)} \cdot \xrightarrow{(10)} x - \mathsf{p}(\mathsf{s}(y - z))$$

Example (cont'd)

- new critical pairs



Example (cont'd)

- new critical pairs

$$x + y \;\xleftarrow{5}\; x + \mathsf{p}(\mathsf{s}(y)) \;\xrightarrow{9}\; \mathsf{p}(x + \mathsf{s}(y))$$

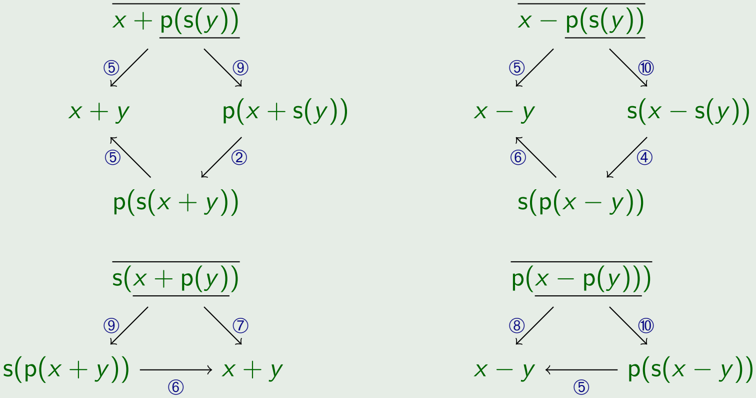
$$x - y \;\xleftarrow{5}\; x - \mathsf{p}(\mathsf{s}(y)) \;\xrightarrow{10}\; \mathsf{s}(x - \mathsf{s}(y))$$

$$\mathsf{s}(\mathsf{p}(x + y)) \;\xleftarrow{9}\; \mathsf{s}(x + \mathsf{p}(y)) \;\xrightarrow{7}\; x + y$$

$$x - y \;\xleftarrow{8}\; \mathsf{p}(x - \mathsf{p}(y)) \;\xrightarrow{10}\; \mathsf{p}(\mathsf{s}(x - y))$$

Example (cont'd)

- new critical pairs



Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$

①	$x + 0 \rightarrow x$	②	$x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$
③	$x - 0 \rightarrow x$	④	$x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$

TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + s(y) \to s(x + y)$
③	$x - 0 \to x$	④	$x - s(y) \to p(x - y)$
⑤	$p(s(x)) \to x$	⑥	$s(p(x)) \to x$
⑦	$s(x + p(y)) \to x + y$	⑧	$p(x - p(y)) \to x - y$
⑨	$x + p(y) \to p(x + y)$	⑩	$x - p(y) \to s(x - y)$

- $\mathcal{S}$ is SN LPO with precedence $+ > s, p$ and $- > s, p$

Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$

TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent

Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + s(y) \to s(x + y)$
③	$x - 0 \to x$	④	$x - s(y) \to p(x - y)$
⑤	$p(s(x)) \to x$	⑥	$s(p(x)) \to x$
⑦	$s(x + p(y)) \to x + y$	⑧	$p(x - p(y)) \to x - y$
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- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent
- $\overset{*}{\underset{\mathcal{S}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

Outline

- Critical Pairs
- Unification
- Completion
  - Example
  - Procedure

Knuth–Bendix Completion Procedure (Simple Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete TRS $\mathcal{R}$ such that $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$

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  if $s' \neq t'$ then

    if $s' > t'$ then

      $\alpha := s' \quad \beta := t'$

    else if $t' > s'$ then

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        $\mathcal{R} := \mathcal{R} \cup \{\alpha \to \beta\}$

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Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure

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1. terminate without failure $\implies$ $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

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Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ $\mathcal{R}$ is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$
2. terminate with failure
3. not terminate (divergence)

Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$f(x, y) \to g(x) \qquad f(x, y) \to h(y)$$

Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad \mathsf{f}(x, y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow \rtimes \to \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow \rtimes \to \mathsf{g}(x)$$

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad \mathsf{f}(x, y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow \rtimes \rightarrow \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow \rtimes \rightarrow \mathsf{g}(x)$$

- no orientation possible $\implies$ failure

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Example

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$$\begin{aligned} \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

Three Possibilities

Knuth-Bendix completion procedure may

3 not terminate (divergence)

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- rewrite rules

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$

Three Possibilities

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- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pair

$$\mathsf{f}(\mathsf{b}) \leftarrow \rtimes \rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$

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Knuth-Bendix completion procedure may

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- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b}$$

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- critical pair

$$\mathsf{f}(\mathsf{b}) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$

Three Possibilities

Knuth-Bendix completion procedure may

3 not terminate (divergence)

Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(h(a)) \to f(b)$$
$$g(a) \to b$$

- LPO with precedence $a > f > g > h > b$
- critical pairs

$$f(b) \leftarrow \rtimes \to g(h(a))$$
$$f(f(b)) \leftarrow \rtimes \to g(h(h(a)))$$

Three Possibilities

Knuth-Bendix completion procedure may

3 not terminate (divergence)

Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow \rtimes \to \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow \rtimes \to \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$

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- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(h(a)) \to f(b)$$
$$g(a) \to b \qquad g(h(h(a))) \to f(f(b))$$

- LPO with precedence $a > f > g > h > b$
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$$f(b) \leftarrow \rtimes \rightarrow g(h(a))$$
$$f(f(b)) \leftarrow \rtimes \rightarrow g(h(h(a)))$$
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Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(a) \to b$$

$$g(h(a)) \to f(b)$$
$$g(h(h(a))) \to f(f(b))$$
$$g(h(h(h(a)))) \to f(f(f(b)))$$
$$\cdots$$

- LPO with precedence $a > f > g > h > b$
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$$f(b) \leftarrow \rtimes \to g(h(a))$$
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Simple Word Problems in Universal Algebras
Donald E. Knuth and Peter Bendix
in: Computational Problems in Abstract Algebra, Pergamon Press, pp. 263 – 297, 1970