Introduction to Term Rewriting

lecture 5

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

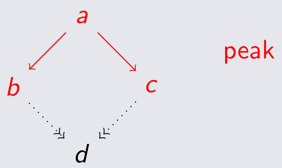
- Critical Pairs
- Unification
- Completion

Newman's Lemma

SN & WCR $\quad\Rightarrow\quad$ CR

Definition (WCR)

$\forall a, b, c$



$\exists d$

Question

how to prove WCR ?

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three peaks

$$\mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})}) \quad \xleftarrow{1} \text{ and } \xrightarrow{2} \quad \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{b})) \qquad \mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{c})$$

parallel redexes

non-critical

$$\overline{\mathsf{f}(\mathsf{a}, \mathsf{g}(\underline{\mathsf{g}(\mathsf{b})}))} \quad \xleftarrow{21} \text{ and } \xrightarrow{\epsilon} \quad \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c})) \qquad \mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{g}(\mathsf{b}))$$

variable overlap

non-critical

$$\overline{\mathsf{f}(\mathsf{a}, \underline{\mathsf{g}(\mathsf{b})})} \quad \xleftarrow{2} \text{ and } \xrightarrow{\epsilon} \quad \mathsf{f}(\mathsf{a},\mathsf{c}) \qquad \mathsf{f}(\mathsf{b},\mathsf{b})$$

overlapping redexes

critical

Definitions

- overlap is triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are rewrite rules without common variables
  2. $p \in \mathcal{P}os_{\mathcal{F}}(\ell_2)$
  3. ℓ_1 and $\ell_2|_p$ are unifiable with most general unifier σ
  4. if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants
- $\ell_2\sigma[r_1\sigma]_p \quad \leftarrow\rtimes\rightarrow \quad r_2\sigma$ critical pair
- critical pair $s \leftarrow\rtimes\rightarrow t$ is convergent if $s \downarrow t$

Critical Pair Lemma (Huet 1980)

TRS is locally confluent $\iff$ all critical pairs are convergent

Example

$$\mathrm{e} \cdot x \to x \qquad x^- \cdot x \to \mathrm{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathrm{e} \cdot u \to u,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, u^- \cdot u \to \mathrm{e},\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (u \cdot v) \cdot w \to u \cdot (v \cdot w),\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $u \cdot z \leftarrow \rtimes \to \mathrm{e} \cdot (u \cdot z)$ convergent
- $\mathrm{e} \cdot z \leftarrow \rtimes \to u^- \cdot (u \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow \rtimes \to (u \cdot v) \cdot (w \cdot z)$ convergent

Theorem (Knuth & Bendix 1970)

terminating TRS is confluent $\iff$ all critical pairs are convergent

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Definition

composition of substitutions σ and τ:

$$\sigma\tau = \{x \mapsto \sigma(x)\tau \mid x \in \mathcal{V}\}$$

Example

$\sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\} \quad \tau = \{x \mapsto \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$

- $\sigma\tau = \{x \mapsto \mathsf{s}(y), y \mapsto \mathsf{s}(0) + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$
- $\tau\sigma = \{x \mapsto \mathsf{s}(0), y \mapsto x + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(x + \mathsf{s}(0)\}$

Lemma

$(\rho\sigma)\tau = \rho(\sigma\tau)$ *for all substitutions* ρ, σ, τ

Definitions

- $\lesssim$ subsumption

  $s \lesssim t \iff \exists\sigma : s\sigma = t$ "s subsumes t" "t is instance of s"

- $\lessdot$ proper subsumption

  $s \lessdot t \iff s \lesssim t \ \wedge\ \neg(t \lesssim s)$

Example

$x + y \lesssim s(y) + s(0)$ $s(x) + y \not\lesssim x + s(0)$ $s(x) + y \lesssim s(x) + x$

Lemma

$\gtrdot$ *is well-founded order on terms*

Definitions

- $\doteq$ literal similarity

  $s \doteq t \iff s \lesssim t \;\wedge\; t \lesssim s$

- variable substitution is substitution from $\mathcal{V}$ to $\mathcal{V}$
- renaming is bijective variable substitution
- terms s and t are variants if $s = t\sigma$ for some renaming σ

Lemma

terms s and t are variants $\iff$ $s \doteq t$

Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \doteq \mathsf{s}(y) + \mathsf{s}(z + 0)$ $\qquad$ $\mathsf{s}(x) + \mathsf{s}(y + 0) \not\doteq \mathsf{s}(x) + \mathsf{s}(x + 0)$

Definition (Unification Problem)

instance: terms s, t

question: $\exists \underbrace{\text{substitution } \sigma}_{\text{unifier}} \colon \; s\sigma = t\sigma$?

Definition

substitution σ is at least as general as τ ($\sigma \lesssim \tau$) if $\exists$ substitution ρ: $\sigma\rho = \tau$

Lemma

$>$ *is well-founded order on substitutions*

Definition

most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

d decomposition

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n),\ E_2}{E_1,\ s_1 \approx t_1,\ \ldots,\ s_n \approx t_n,\ E_2}$$

t removal of trivial equations $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx x,\ E_2}{E_1,\ E_2}$$

v variable elimination $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx t,\ E_2}{(E_1,\ E_2)\sigma} \quad \text{and} \quad \frac{E_1,\ t \approx x,\ E_2}{(E_1,\ E_2)\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

Example

$$x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)$$

$$\mathsf{d} \Downarrow$$

$$x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x$$

$$\mathsf{v} \Downarrow \ x \mapsto \mathsf{s}(z)$$

$$0 + \mathsf{s}(y) \approx 0 + \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$0 \approx 0,\ \mathsf{s}(y) \approx \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$\mathsf{s}(y) \approx \mathsf{s}(z)$$

$$\mathsf{d} \Downarrow$$

$$y \approx z$$

$$\mathsf{v} \Downarrow \ y \mapsto z$$

$$\square$$

mgu $\{x \mapsto \mathsf{s}(z), y \mapsto z\}$

Theorem

- *there are no infinite derivations*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots$$

- *if s and t are unifiable then for every maximal derivation*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots \Rightarrow_{\sigma_n} E_n$$

  - $E_n = \square$
  - $\sigma_1\sigma_2\sigma_3 \cdots \sigma_n$ *is mgu of s and t*

Optional Failure Rules

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx g(t_1,\ldots,t_m),\ E_2}{\bot} \qquad \frac{E_1,\ x \approx t,\ E_2}{\bot} \quad \frac{E_1,\ t \approx x,\ E_2}{\bot}$$

if $x \in \mathcal{V}\mathrm{ar}(t)$

Outline

- Critical Pairs
- Unification
- Completion
  - Example
  - Procedure

Example

TRS $\mathcal{R}$

$$
\begin{array}{rrcl@{\qquad}rrcl}
① & x + 0 & \to & x & ② & x - 0 & \to & x \\
③ & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
⑦ & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y & ⑧ & \mathsf{p}(x - \mathsf{p}(y)) & \to & x - y
\end{array}
$$

- SN ? (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{⑥}\ x + y \qquad \xrightarrow{③}\ \mathsf{s}(x + \mathsf{p}(y)) \qquad x + y \xleftarrow{⑦} \mathsf{s}(x + \mathsf{p}(y))$$

$$\overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{⑥}\ x - y \qquad \xrightarrow{④}\ \mathsf{p}(x - \mathsf{p}(y)) \qquad x - y \xleftarrow{⑧} \mathsf{p}(x - \mathsf{p}(y))$$

$$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \quad \xrightarrow{⑥}\ \mathsf{p}(x) \qquad \xrightarrow{⑤}\ \mathsf{p}(x) \qquad \mathsf{p}(x) = \mathsf{p}(x)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \quad \xrightarrow{⑤}\ \mathsf{s}(x) \qquad \xrightarrow{⑥}\ \mathsf{s}(x) \qquad \mathsf{s}(x) = \mathsf{s}(x)$$

Example (cont'd)

- added rewrite rules

$$⑦\quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad ⑧\quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

  preserve termination and do not change $\overset{*}{\longleftrightarrow}$

- new critical pairs

$\overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})}$: $\xrightarrow{⑦}$ $\mathsf{p}(x + y)$ and $\xrightarrow{⑤}$ $x + \mathsf{p}(y)$; $\mathsf{p}(x + y) \xleftarrow{⑨} x + \mathsf{p}(y)$

$\overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})}$: $\xrightarrow{⑤}$ $\mathsf{s}(x + y)$ and $\xrightarrow{⑦}$ $x + \mathsf{s}(y)$; $\mathsf{s}(x + y) \xleftarrow{②} x + \mathsf{s}(y)$

$\overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})}$: $\xrightarrow{⑧}$ $\mathsf{s}(x - y)$ and $\xrightarrow{⑥}$ $x - \mathsf{p}(y)$; $\mathsf{s}(x - y) \xleftarrow{⑩} x - \mathsf{p}(y)$

$\overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})}$: $\xrightarrow{⑤}$ $\mathsf{p}(x - y)$ and $\xrightarrow{⑧}$ $x - \mathsf{s}(y)$; $\mathsf{p}(x - y) \xleftarrow{④} x - \mathsf{s}(y)$

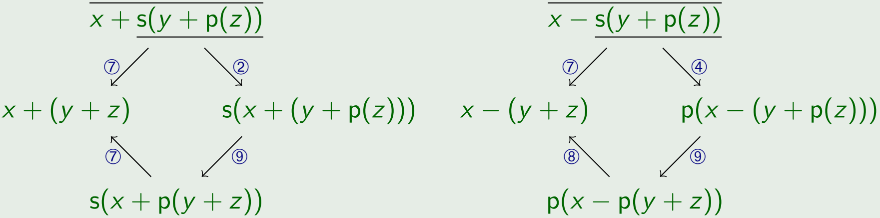
Example (cont'd)

- added rewrite rules

$$⑨\quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩\quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

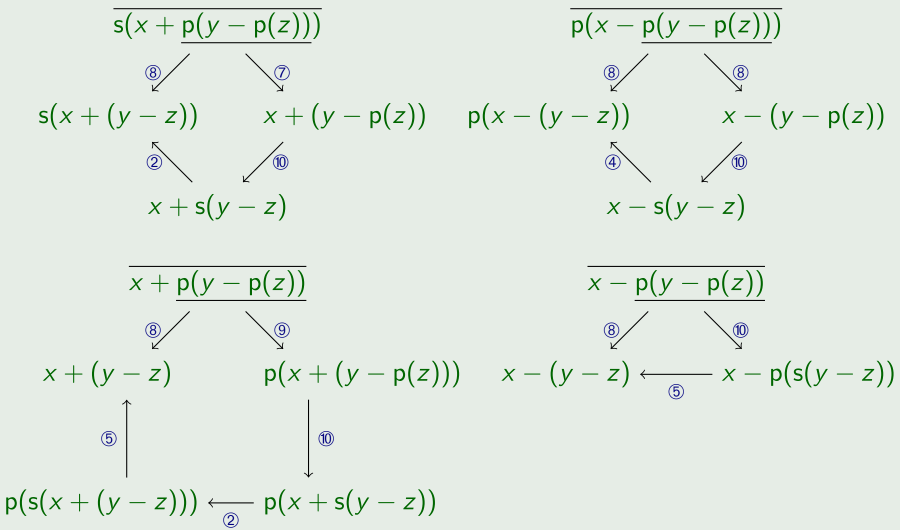
  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- new critical pairs



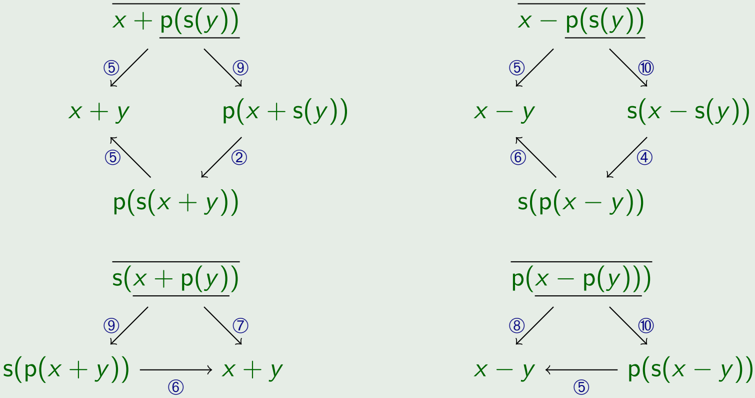
Example (cont'd)

- new critical pairs



Example (cont'd)

- new critical pairs



Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent
- $\overset{*}{\underset{\mathcal{S}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

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Knuth–Bendix Completion Procedure (Simple Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete TRS $\mathcal{R}$ such that $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

$\mathcal{R} := \varnothing \qquad C := \mathcal{E}$

while $C \neq \varnothing$ do

  choose $s \approx t \in C \qquad C := C \setminus \{s \approx t\}$

  compute $\mathcal{R}$-normal forms s' and t' of s and t

  if $s' \neq t'$ then

    if $s' > t'$ then

      $\alpha := s' \qquad \beta := t'$

    else if $t' > s'$ then

      $\alpha := t' \qquad \beta := s'$

    else

      failure

    $\mathcal{R} := \mathcal{R} \cup \{\alpha \to \beta\}$

    $C := C \cup \{e \in \mathrm{CP}(\mathcal{R}) \mid \alpha \to \beta$ was used to generate $e\}$

Invariants

1. $\mathcal{R} \subseteq >$
2. $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*} \cup \xleftrightarrow[C]{*}$
3. equations in $\mathsf{CP}(\mathcal{R}) \setminus C$ are convergent with respect to $\mathcal{R}$

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ $\mathcal{R}$ is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$
2. terminate with failure
3. not terminate (divergence)

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$f(x, y) \rightarrow g(x) \qquad f(x, y) \rightarrow h(y)$$

- two critical pairs

$$g(x) \leftarrow \rtimes \rightarrow h(y) \qquad h(y) \leftarrow \rtimes \rightarrow g(x)$$

- no orientation possible $\Longrightarrow$ failure

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(a) \to b$$

$$g(h(a)) \to f(b)$$
$$g(h(h(a))) \to f(f(b))$$
$$g(h(h(h(a)))) \to f(f(f(b)))$$
$$\cdots$$

- LPO with precedence $a > f > g > h > b$
- critical pairs

$$f(b) \leftarrow \rtimes \rightarrow g(h(a))$$
$$f(f(b)) \leftarrow \rtimes \rightarrow g(h(h(a)))$$
$$f(f(f(b))) \leftarrow \rtimes \rightarrow g(h(h(h(a))))$$

Simple Word Problems in Universal Algebras
Donald E. Knuth and Peter Bendix
in: Computational Problems in Abstract Algebra, Pergamon Press, pp. 263 – 297, 1970