Introduction to Term Rewriting
lecture 5

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

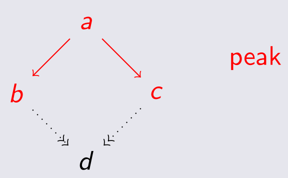
- Critical Pairs
- Unification
- Completion

Newman's Lemma

SN & WCR $\Rightarrow$ CR

Definition (WCR)

$\forall a, b, c$



$\exists d$

Question

how to prove WCR ?

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three peaks

$\mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})})$	$\overline{\mathsf{f}(\mathsf{a}, \underline{\mathsf{g}(\mathsf{b})})})$	$\overline{\mathsf{f}(\mathsf{a}, \underline{\mathsf{g}(\mathsf{b})})}$
$\swarrow 1$ | $2 \searrow$	$\swarrow 21$ | $\epsilon \searrow$	$\swarrow 2$ | $\epsilon \searrow$
f(c,g(b)) | f(g(b),c)	f(a,g(c)) | f(g(b),g(b))	f(a,c) | f(b,b)
parallel redexes	variable overlap	overlapping redexes
non-critical	non-critical	critical

Definitions

- overlap is triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are rewrite rules without common variables
  2. $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$
  3. ℓ_1 and $\ell_2|_p$ are unifiable with most general unifier σ
  4. if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants
- $\ell_2\sigma[r_1\sigma]_p \quad \leftarrow\rtimes\rightarrow \quad r_2\sigma$ critical pair
- critical pair $s \leftarrow\rtimes\rightarrow t$ is convergent if $s \downarrow t$

Critical Pair Lemma (Huet 1980)

TRS is locally confluent $\iff$ all critical pairs are convergent

Example

$$e \cdot x \to x \qquad x^- \cdot x \to e \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, e \cdot u \to u,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z)\,\rangle$
- $\langle\, u^- \cdot u \to e,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z)\,\rangle$
- $\langle\, (u \cdot v) \cdot w \to u \cdot (v \cdot w),\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z)\,\rangle$

critical pairs

- $u \cdot z \leftarrow\!\rtimes\!\!\!\ltimes\!\rightarrow e \cdot (u \cdot z)$ convergent
- $e \cdot z \leftarrow\!\rtimes\!\!\!\ltimes\!\rightarrow u^- \cdot (u \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\!\rtimes\!\!\!\ltimes\!\rightarrow (u \cdot v) \cdot (w \cdot z)$ convergent

Theorem (Knuth & Bendix 1970)

terminating TRS is confluent $\iff$ all critical pairs are convergent

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Definition

composition of substitutions σ and τ:

$$\sigma\tau = \{x \mapsto \sigma(x)\tau \mid x \in \mathcal{V}\}$$

Example

$\sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\} \qquad \tau = \{x \mapsto \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$

- $\sigma\tau = \{x \mapsto \mathsf{s}(y), y \mapsto \mathsf{s}(0) + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$
- $\tau\sigma = \{x \mapsto \mathsf{s}(0), y \mapsto x + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(x + \mathsf{s}(0))\}$

Lemma

$(\rho\sigma)\tau = \rho(\sigma\tau)$ *for all substitutions* ρ, σ, τ

Definitions

- $\lesssim$ subsumption

  $s \lesssim t \iff \exists\sigma\colon s\sigma = t$ "s subsumes t" "t is instance of s"

- $<$ proper subsumption

  $s < t \iff s \lesssim t \ \wedge\ \neg(t \lesssim s)$

Example

$x + y \lesssim \mathsf{s}(y) + \mathsf{s}(0) \qquad \mathsf{s}(x) + y \not\lesssim x + \mathsf{s}(0) \qquad \mathsf{s}(x) + y \lesssim \mathsf{s}(x) + x$

Lemma

$>$ *is well-founded order on terms*

Definitions

- $\doteq$ literal similarity
  $s \doteq t \iff s \leq t \wedge t \leq s$
- variable substitution is substitution from $\mathcal{V}$ to $\mathcal{V}$
- renaming is bijective variable substitution
- terms s and t are variants if $s = t\sigma$ for some renaming σ

Lemma

terms s and t are variants $\iff$ $s \doteq t$

Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \doteq \mathsf{s}(y) + \mathsf{s}(z + 0)$ $\quad$ $\mathsf{s}(x) + \mathsf{s}(y + 0) \not\doteq \mathsf{s}(x) + \mathsf{s}(x + 0)$

Definition (Unification Problem)

instance: terms s, t

question: $\exists\ \underbrace{\text{substitution } \sigma}_{\text{unifier}}:\ s\sigma = t\sigma$?

Definition

substitution σ is at least as general as τ ($\sigma \leq \tau$) if $\exists$ substitution ρ: $\sigma\rho = \tau$

Lemma

$>$ *is well-founded order on substitutions*

Definition

most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

d decomposition

$$\frac{E_1,\ f(s_1,\dots,s_n) \approx f(t_1,\dots,t_n),\ E_2}{E_1,\ s_1 \approx t_1,\ \dots,\ s_n \approx t_n,\ E_2}$$

t removal of trivial equations $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx x,\ E_2}{E_1,\ E_2}$$

v variable elimination $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx t,\ E_2}{(E_1,\ E_2)\sigma} \quad \text{and} \quad \frac{E_1,\ t \approx x,\ E_2}{(E_1,\ E_2)\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

Example

$x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)$

d $\Downarrow$

$x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x$

v $\Downarrow$ $x \mapsto \mathsf{s}(z)$

$0 + \mathsf{s}(y) \approx 0 + \mathsf{s}(z)$

d $\Downarrow$

$0 \approx 0,\ \mathsf{s}(y) \approx \mathsf{s}(z)$ mgu $\{x \mapsto \mathsf{s}(z), y \mapsto z\}$

d $\Downarrow$

$\mathsf{s}(y) \approx \mathsf{s}(z)$

d $\Downarrow$

$y \approx z$

v $\Downarrow$ $y \mapsto z$

$\square$

Theorem

- *there are no infinite derivations*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots$$

- *if s and t are unifiable then for every maximal derivation*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots \Rightarrow_{\sigma_n} E_n$$

  - $E_n = \square$
  - $\sigma_1\sigma_2\sigma_3\cdots\sigma_n$ *is mgu of s and t*

Optional Failure Rules

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx g(t_1,\ldots,t_m),\ E_2}{\bot} \qquad \frac{E_1,\ x \approx t,\ E_2}{\bot} \quad \frac{E_1,\ t \approx x,\ E_2}{\bot}$$

if $x \in \mathcal{V}\text{ar}(t)$

Outline

- Critical Pairs
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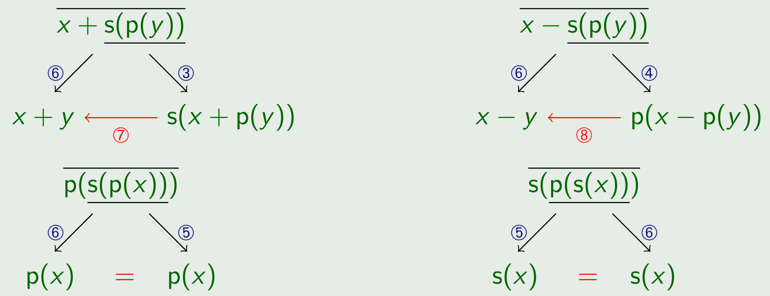
- Completion
  - Example
  - Procedure

Example

TRS $\mathcal{R}$

①	$x + 0$	$\rightarrow$	x	②	$x - 0$	$\rightarrow$	x
③	$x + s(y)$	$\rightarrow$	$s(x + y)$	④	$x - s(y)$	$\rightarrow$	$p(x - y)$
⑤	$p(s(x))$	$\rightarrow$	x	⑥	$s(p(x))$	$\rightarrow$	x
⑦	$s(x + p(y))$	$\rightarrow$	$x + y$	⑧	$p(x - p(y))$	$\rightarrow$	$x - y$

- SN ? (e.g.) LPO with precedence $+ > s$ and $- > p$
- WCR ? 4 critical pairs

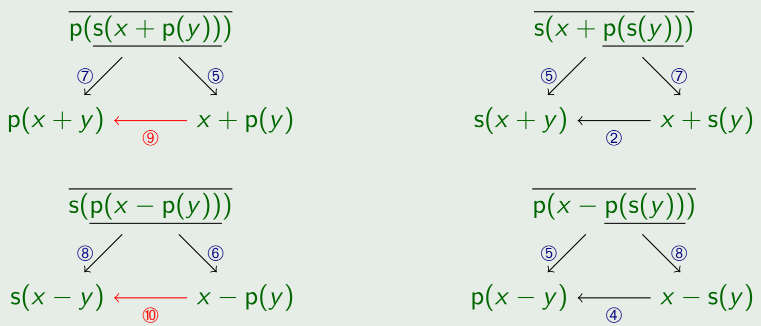


Example (cont'd)

- added rewrite rules

  ⑦ $s(x + p(y)) \rightarrow x + y$ ⑧ $p(x - p(y)) \rightarrow x - y$

  preserve termination and do not change $\overset{*}{\longleftrightarrow}$
- new critical pairs



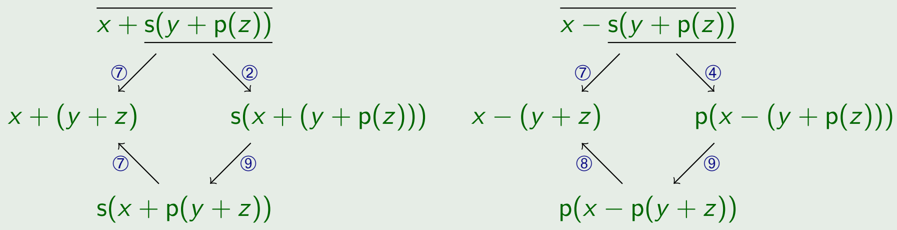
Example (cont'd)

- added rewrite rules

$$⑨\quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩\quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

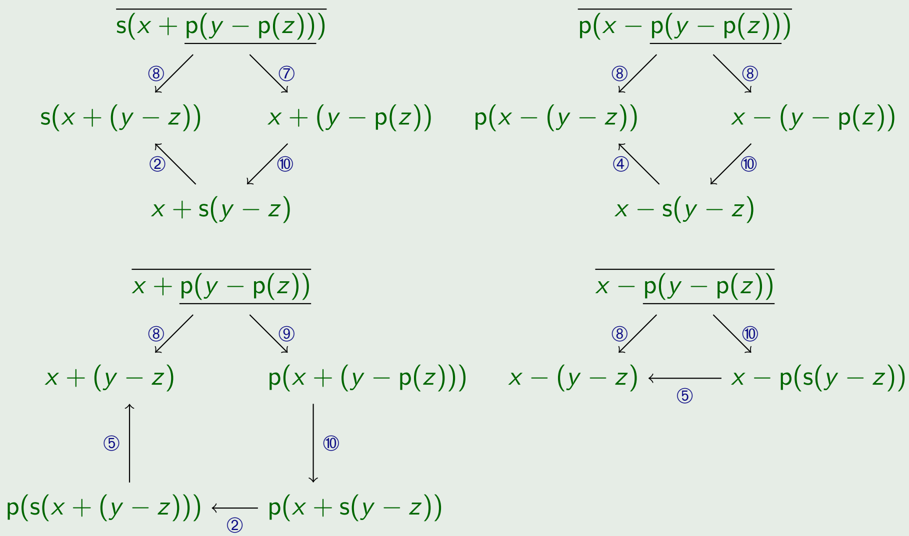
  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- new critical pairs



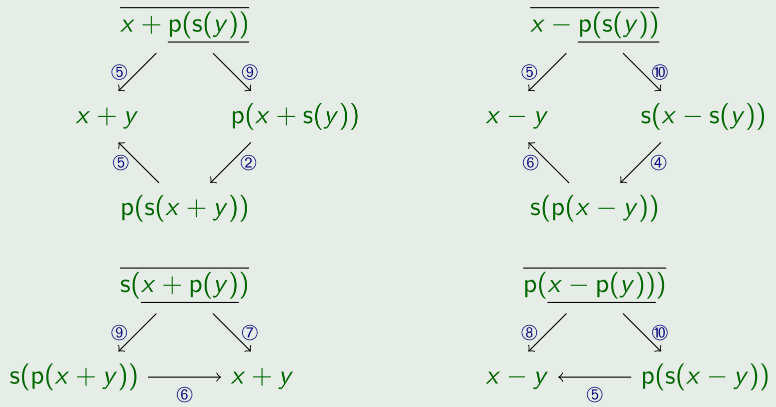
Example (cont'd)

- new critical pairs



Example (cont'd)

- new critical pairs



Example (cont'd)

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent
- $\overset{*}{\underset{\mathcal{S}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

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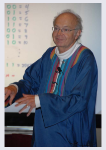
Knuth–Bendix Completion Procedure (Simple Version)



input ES $\mathcal{E}$ and reduction order $>$

output complete TRS $\mathcal{R}$ such that $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$

$\mathcal{R} := \varnothing \quad C := \mathcal{E}$

while $C \neq \varnothing$ do
 choose $s \approx t \in C \quad C := C \setminus \{s \approx t\}$
 compute $\mathcal{R}$-normal forms s' and t' of s and t
 if $s' \neq t'$ then
  if $s' > t'$ then
   $\alpha := s' \quad \beta := t'$
  else if $t' > s'$ then
   $\alpha := t' \quad \beta := s'$
  else
   failure
  $\mathcal{R} := \mathcal{R} \cup \{\alpha \to \beta\}$
  $C := C \cup \{e \in \mathrm{CP}(\mathcal{R}) \mid \alpha \to \beta$ was used to generate $e\}$

Invariants

1. $\mathcal{R} \subseteq >$
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} \cup \overset{*}{\underset{\mathcal{C}}{\longleftrightarrow}}$
3. equations in $\mathsf{CP}(\mathcal{R}) \setminus \mathcal{C}$ are convergent with respect to $\mathcal{R}$

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ $\mathcal{R}$ is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$
2. terminate with failure
3. not terminate (divergence)

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad \mathsf{f}(x, y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow\rtimes\rightarrow \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow\rtimes\rightarrow \mathsf{g}(x)$$

- no orientation possible $\implies$ failure

Three Possibilities

Knuth-Bendix completion procedure may

3 not terminate (divergence)

Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{a}) \to \mathsf{b}$$

$$\mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b}))$$
$$\mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a})))) \to \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b})))$$
$$\cdots$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a}))))$$

Simple Word Problems in Universal Algebras
Donald E. Knuth and Peter Bendix
in: Computational Problems in Abstract Algebra, Pergamon Press, pp. 263 – 297, 1970