ISR 2010

Introduction to Term Rewriting

lecture 5

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Critical Pairs
- Unification
- Completion

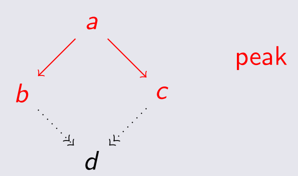
Newman's Lemma

SN & WCR $\Rightarrow$ CR

Definition (WCR)

$\forall a, b, c$

$\exists d$



Question

how to prove WCR ?

Example

$$\mathsf{f}(\mathsf{a},\mathsf{g}(x)) \to \mathsf{f}(x,x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three peaks

parallel redexes	variable overlap	overlapping redexes
$\mathsf{f}(\underline{\mathsf{g}(\mathsf{b})},\overline{\mathsf{g}(\mathsf{b})})$	$\overline{\mathsf{f}(\mathsf{a},\mathsf{g}(\underline{\mathsf{g}(\mathsf{b})}))}$	$\overline{\mathsf{f}(\mathsf{a},\underline{\mathsf{g}(\mathsf{b})})}$
1 ↙ ↘ 2	21 ↙ ↘ ϵ	2 ↙ ↘ ϵ
f(c,g(b)) f(g(b),c)	f(a,g(c)) f(g(b),g(b))	f(a,c) f(b,b)
non-critical	non-critical	critical

Definitions

- overlap is triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are rewrite rules without common variables
  2. $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$
  3. ℓ_1 and $\ell_2|_p$ are unifiable with most general unifier σ
  4. if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants
- $\ell_2\sigma[r_1\sigma]_p \quad \leftarrow\rtimes\rightarrow \quad r_2\sigma$ critical pair
- critical pair $s \leftarrow\rtimes\rightarrow t$ is convergent if $s \downarrow t$

Critical Pair Lemma (Huet 1980)

TRS is locally confluent $\iff$ all critical pairs are convergent

Example

$$\mathsf{e}\cdot x \to x \qquad x^- \cdot x \to \mathsf{e} \qquad (x\cdot y)\cdot z \to x\cdot(y\cdot z)$$

overlaps

- $\langle\, \mathsf{e}\cdot u \to u,\ 1,\ (x\cdot y)\cdot z \to x\cdot(y\cdot z)\,\rangle$
- $\langle\, u^-\cdot u \to \mathsf{e},\ 1,\ (x\cdot y)\cdot z \to x\cdot(y\cdot z)\,\rangle$
- $\langle\, (u\cdot v)\cdot w \to u\cdot(v\cdot w),\ 1,\ (x\cdot y)\cdot z \to x\cdot(y\cdot z)\,\rangle$

critical pairs

- $u\cdot z \leftarrow\rtimes\rightarrow \mathsf{e}\cdot(u\cdot z)$ convergent
- $\mathsf{e}\cdot z \leftarrow\rtimes\rightarrow u^-\cdot(u\cdot z)$ not convergent
- $(u\cdot(v\cdot w))\cdot z \leftarrow\rtimes\rightarrow (u\cdot v)\cdot(w\cdot z)$ convergent

Theorem (Knuth & Bendix 1970)

terminating TRS is confluent $\iff$ all critical pairs are convergent

Outline

Definition

composition of substitutions σ and τ:

$$\sigma\tau = \{x \mapsto \sigma(x)\tau \mid x \in \mathcal{V}\}$$

Example

$\sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\}$ $\quad \tau = \{x \mapsto \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$

- $\sigma\tau = \{x \mapsto \mathsf{s}(y), y \mapsto \mathsf{s}(0) + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$
- $\tau\sigma = \{x \mapsto \mathsf{s}(0), y \mapsto x + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(x + \mathsf{s}(0))\}$

Lemma

$(\rho\sigma)\tau = \rho(\sigma\tau)$ for all substitutions ρ, σ, τ

Definitions

- $\lesssim$ subsumption

  $s \lesssim t \iff \exists\sigma\colon s\sigma = t$ "s subsumes t" "t is instance of s"

- $\lessdot$ proper subsumption

  $s \lessdot t \iff s \lesssim t \wedge \neg(t \lesssim s)$

Example

$x + y \lesssim \mathsf{s}(y) + \mathsf{s}(0)$ $\quad \mathsf{s}(x) + y \not\lesssim x + \mathsf{s}(0)$ $\quad \mathsf{s}(x) + y \lesssim \mathsf{s}(x) + x$

Lemma

$\gtrdot$ is well-founded order on terms

Definitions

- $\doteq$ literal similarity

  $s \doteq t \iff s \lesssim t \wedge t \lesssim s$

- variable substitution is substitution from $\mathcal{V}$ to $\mathcal{V}$
- renaming is bijective variable substitution
- terms s and t are variants if $s = t\sigma$ for some renaming σ

Lemma

terms s and t are variants $\iff s \doteq t$

Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \doteq \mathsf{s}(y) + \mathsf{s}(z + 0)$ $\quad \mathsf{s}(x) + \mathsf{s}(y + 0) \not\doteq \mathsf{s}(x) + \mathsf{s}(x + 0)$

Definition (Unification Problem)

instance: terms s, t

question: $\exists \underbrace{\text{substitution } \sigma}_{\text{unifier}}\colon s\sigma = t\sigma$?

Definition

substitution σ is at least as general as τ ($\sigma \lesssim \tau$) if $\exists$ substitution $\rho\colon \sigma\rho = \tau$

Lemma

$\gtrdot$ is well-founded order on substitutions

Definition

most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

d decomposition

$$\frac{E_1,\ f(s_1,\dots,s_n) \approx f(t_1,\dots,t_n),\ E_2}{E_1,\ s_1 \approx t_1,\ \dots,\ s_n \approx t_n,\ E_2}$$

t removal of trivial equations $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx x,\ E_2}{E_1,\ E_2}$$

v variable elimination $(x \in \mathcal{V})$

$$\frac{E_1,\ x \approx t,\ E_2}{(E_1,\ E_2)\sigma} \quad \text{and} \quad \frac{E_1,\ t \approx x,\ E_2}{(E_1,\ E_2)\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

Example

$$x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)$$
$$\mathsf{d} \Downarrow$$
$$x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x$$
$$\mathsf{v} \Downarrow \ x \mapsto \mathsf{s}(z)$$
$$0 + \mathsf{s}(y) \approx 0 + \mathsf{s}(z)$$
$$\mathsf{d} \Downarrow$$
$$0 \approx 0,\ \mathsf{s}(y) \approx \mathsf{s}(z)$$
$$\mathsf{d} \Downarrow$$
$$\mathsf{s}(y) \approx \mathsf{s}(z)$$
$$\mathsf{d} \Downarrow$$
$$y \approx z$$
$$\mathsf{v} \Downarrow \ y \mapsto z$$
$$\square$$

mgu $\{x \mapsto \mathsf{s}(z), y \mapsto z\}$

Theorem

- *there are no infinite derivations*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots$$

- *if s and t are unifiable then for every maximal derivation*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots \Rightarrow_{\sigma_n} E_n$$

  - $E_n = \square$
  - $\sigma_1\sigma_2\sigma_3\cdots\sigma_n$ *is mgu of s and t*

Optional Failure Rules

$$\frac{E_1,\ f(s_1,\dots,s_n) \approx g(t_1,\dots,t_m),\ E_2}{\bot} \qquad \frac{E_1,\ x \approx t,\ E_2}{\bot} \quad \frac{E_1,\ t \approx x,\ E_2}{\bot}$$

if $x \in \mathcal{V}\mathrm{ar}(t)$

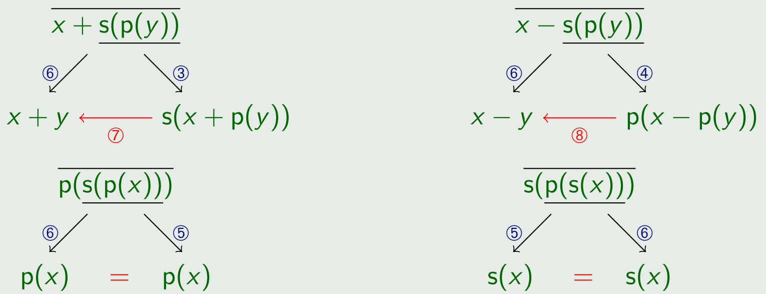
Outline

Example

TRS $\mathcal{R}$

①	$x + 0 \rightarrow x$	②	$x - 0 \rightarrow x$
③	$x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \rightarrow x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \rightarrow x - y$

- SN ? (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

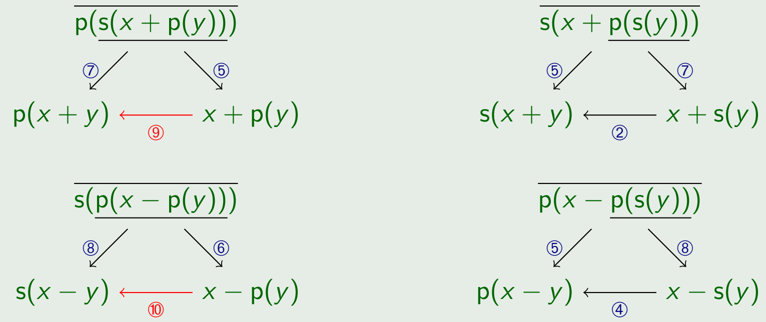


Example (cont'd)

- added rewrite rules

  ⑦ $\mathsf{s}(x + \mathsf{p}(y)) \rightarrow x + y$ ⑧ $\mathsf{p}(x - \mathsf{p}(y)) \rightarrow x - y$

  preserve termination and do not change $\overset{*}{\longleftrightarrow}$
- new critical pairs

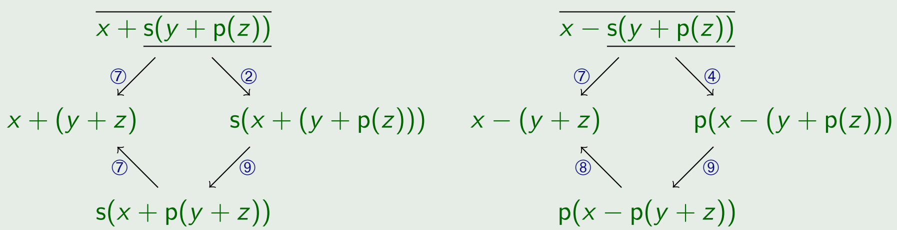


Example (cont'd)

- added rewrite rules

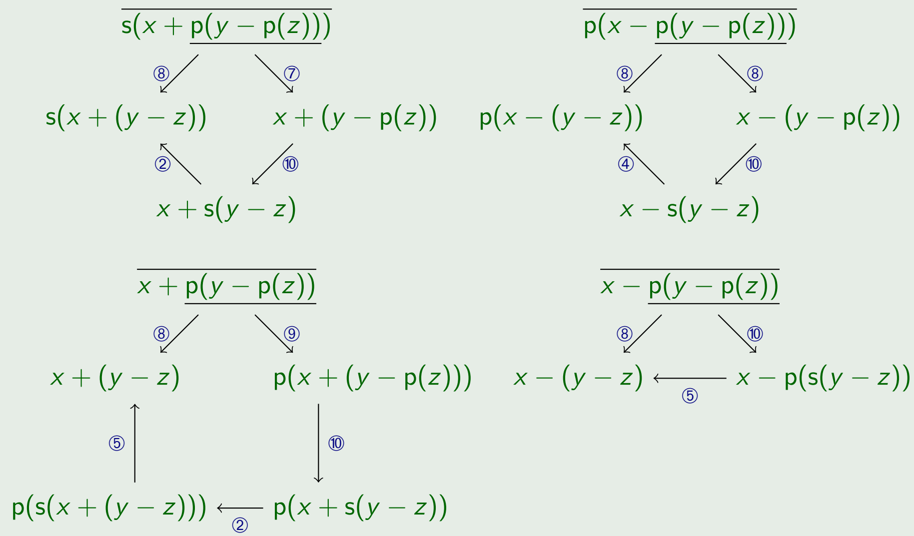
  ⑨ $x + \mathsf{p}(y) \rightarrow \mathsf{p}(x + y)$ ⑩ $x - \mathsf{p}(y) \rightarrow \mathsf{s}(x - y)$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)
- new critical pairs



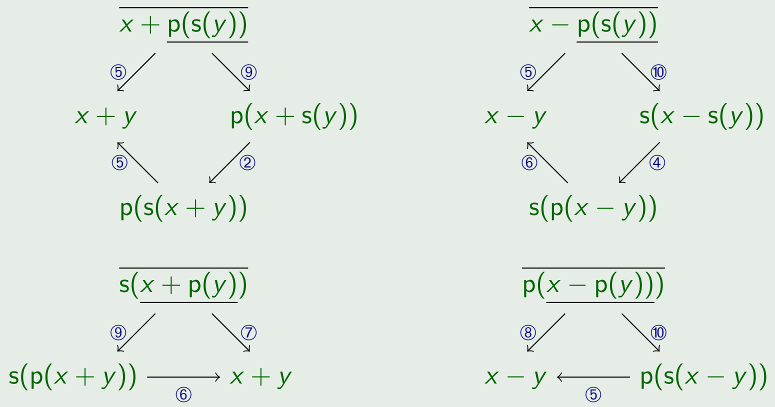
Example (cont'd)

- new critical pairs



Example (cont'd)

- new critical pairs



Example (cont'd)

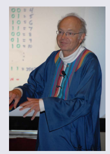
TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent
- $\xleftrightarrow[\mathcal{S}]{*} = \xleftrightarrow[\mathcal{R}]{*}$

Outline

Knuth–Bendix Completion Procedure (Simple Version)



input ES $\mathcal{E}$ and reduction order $>$
output complete TRS $\mathcal{R}$ such that $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$

```
R := ∅     C := E
while C ≠ ∅ do
    choose s ≈ t ∈ C     C := C \ {s ≈ t}
    compute R-normal forms s′ and t′ of s and t
    if s′ ≠ t′ then
        if s′ > t′ then
            α := s′     β := t′
        else if t′ > s′ then
            α := t′     β := s′
        else
            failure
        R := R ∪ {α → β}
        C := C ∪ {e ∈ CP(R) | α → β was used to generate e}
```

Invariants

1. $\mathcal{R} \subseteq >$
2. $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*} \cup \xleftrightarrow[C]{*}$
3. equations in $\mathrm{CP}(\mathcal{R}) \setminus C$ are convergent with respect to $\mathcal{R}$

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ $\mathcal{R}$ is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$
2. terminate with failure
3. not terminate (divergence)

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x,y) \to \mathsf{g}(x) \qquad \mathsf{f}(x,y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow\!\rtimes\!\rightarrow \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow\!\rtimes\!\rightarrow \mathsf{g}(x)$$

- no orientation possible $\implies$ failure

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b}))$$
$$\mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a})))) \to \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b})))$$
$$\cdots$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow\!\rtimes\!\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow\!\rtimes\!\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \leftarrow\!\rtimes\!\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a}))))$$

Simple Word Problems in Universal Algebras
Donald E. Knuth and Peter Bendix
in: Computational Problems in Abstract Algebra, Pergamon Press, pp. 263 – 297, 1970