Introduction to Term Rewriting

lecture 6

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Efficient Completion
- Cola Gene Puzzle
- Abstract Completion
- Proof Orders
- Critical Pair Criteria
- Further Reading

Example

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

Example

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

$$
\begin{array}{llcl}
① & x + 0 & \to & x \\
③ & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x \\
⑦ & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y \\
⑨ & x + \mathsf{p}(y) & \to & \mathsf{p}(x + y)
\end{array}
\qquad
\begin{array}{llcl}
② & x - 0 & \to & x \\
④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
⑧ & \mathsf{p}(x - \mathsf{p}(y)) & \to & x - y \\
⑩ & x - \mathsf{p}(y) & \to & \mathsf{s}(x - y)
\end{array}
$$

Example

TRS $\mathcal{R} = \{$①, ②, ③, ④, ⑤, ⑥$\}$ TRS $\mathcal{S} = \{$①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩$\}$

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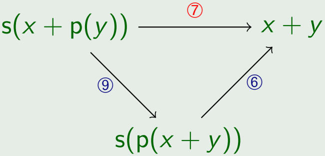
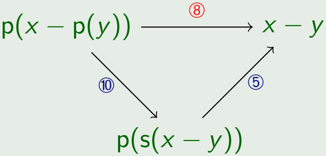
rewrite rules ⑦ and ⑧ are redundant:





Example

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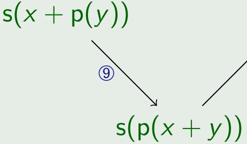

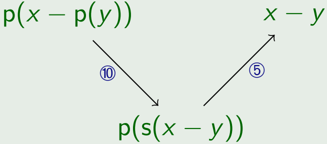

Observation

- less rewrite rules $\Longrightarrow$ less critical pairs
- TRS without redundancy $=$ reduced TRS

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Definition

TRS $\mathcal{R}$ is reduced if for all $\ell \to r \in \mathcal{R}$

1. r is normal form with respect to $\mathcal{R}$

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Definition

TRS $\mathcal{R}$ is reduced if for all $\ell \to r \in \mathcal{R}$

1. r is normal form with respect to $\mathcal{R}$
2. ℓ is normal form with respect to $\mathcal{R} \setminus \{\ell \to r\}$

Example

TRS $\mathcal{R} = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

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⑤	$p(s(x)) \rightarrow x$	⑥	$s(p(x)) \rightarrow x$
⑦	$s(x + p(y)) \rightarrow x + y$	⑧	$p(x - p(y)) \rightarrow x - y$
⑨	$x + p(y) \rightarrow p(x + y)$	⑩	$x - p(y) \rightarrow s(x - y)$

- $\mathcal{R}$ is reduced

Example

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- $\mathcal{R}$ is reduced
- $\mathcal{S}$ is not reduced

simplification after completion

Theorem

$\forall$ *complete TRS* $\mathcal{R}$ $\exists$ *complete reduced TRS* $\mathcal{S}$ *such that* $\overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{S}}{\longleftrightarrow}}$

simplification after completion

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Proof Sketch (construction)

1. $\mathcal{R}' = \{\, \ell \to r{\downarrow_{\mathcal{R}}} \mid \ell \to r \in \mathcal{R} \,\}$

simplification after completion

Theorem

∀ complete TRS $\mathcal{R}$ ∃ complete reduced TRS $\mathcal{S}$ such that $\xleftrightarrow[\mathcal{R}]{*} \; = \; \xleftrightarrow[\mathcal{S}]{*}$

Proof Sketch (construction)

1. $\mathcal{R}' = \{\, \ell \to r{\downarrow_{\mathcal{R}}} \mid \ell \to r \in \mathcal{R} \,\}$
2. $\mathcal{S} \;= \{\, \ell \to r \in \mathcal{R}' \mid \ell \in \mathsf{NF}(\,\mathcal{R}' \setminus \{\ell \to r\}\,) \,\}$

simplification after completion

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more efficient: simplification during completion

Knuth-Bendix Completion Procedure (More Efficient Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete reduced TRS $\mathcal{R}$ such that $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{\mathcal{R}}{\longleftrightarrow}}$

$\mathcal{R} := \varnothing \quad C := \mathcal{E}$

while $C \neq \varnothing$ do

  choose $s \approx t \in C \quad C := C \setminus \{s \approx t\} \quad s' := s{\downarrow_{\mathcal{R}}} \quad t' := t{\downarrow_{\mathcal{R}}}$

  if $s' \neq t'$ then

    if $s' > t'$ then $\alpha := s' \quad \beta := t'$

    else if $t' > s'$ then $\alpha := t' \quad \beta := s'$

    else *failure*

    $\mathcal{R}' := \mathcal{R} \cup \{\alpha \to \beta\}$

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    $\mathcal{R}' := \mathcal{R} \cup \{\alpha \to \beta\}$

    for all $\ell \to r \in \mathcal{R}$ do

      $\mathcal{R}' := \mathcal{R}' \setminus \{\ell \to r\} \quad \ell' := \ell{\downarrow_{\mathcal{R}'}} \quad r' := r{\downarrow_{\mathcal{R}'}}$

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      if $\ell = \ell'$ then $\mathcal{R}' := \mathcal{R}' \cup \{\ell' \to r'\}$ else $C := C \cup \{\ell' \approx r'\}$

Knuth-Bendix Completion Procedure (More Efficient Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete reduced TRS $\mathcal{R}$ such that $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{R}]{*}$

$\mathcal{R} := \varnothing \quad C := \mathcal{E}$

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      if $\ell = \ell'$ then $\mathcal{R}' := \mathcal{R}' \cup \{\ell' \to r'\}$ else $C := C \cup \{\ell' \approx r'\}$

    $\mathcal{R} := \mathcal{R}'$

    $C := C \cup \{e \in \mathrm{CP}(\mathcal{R}) \mid \alpha \to \beta$ was used to generate $e\}$

Example

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\approx \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\approx \mathsf{b} \end{aligned}$$

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- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$

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- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- orient: $\mathsf{f}(\mathsf{f}(x)) >_{\mathsf{lpo}} \mathsf{g}(x)$

Example

$$f(f(x)) \rightarrow g(x)$$

$$g(a) \approx b$$

- LPO with precedence $f > g > b > a$
- orient: $f(f(x)) >_{lpo} g(x)$

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- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce: $\mathsf{f}(\mathsf{g}(x)) \leftarrow \mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \to \mathsf{g}(\mathsf{f}(x))$ critical pair

Example

$$\mathsf{f}(\mathsf{g}(x)) \approx \mathsf{g}(\mathsf{f}(x))$$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$
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- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
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Example

$$f(g(x)) \approx g(f(x))$$

$$f(f(x)) \to g(x)$$
$$g(a) \to b$$

- LPO with precedence $f > g > b > a$
- orient: $f(g(x)) >_{lpo} g(f(x))$

Example

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- orient: $f(g(x)) >_{lpo} g(f(x))$

Example

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- deduce: $f(g(f(x))) \leftarrow f(f(g(x))) \to g(g(x))$ critical pair

Example

$$\mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \approx \mathsf{g}(\mathsf{g}(x))$$

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce: $\mathsf{f}(\mathsf{b}) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \rightarrow \mathsf{g}(\mathsf{f}(\mathsf{a}))$ critical pair

Example

$$\begin{aligned} \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) &\approx \mathsf{g}(\mathsf{g}(x)) \\ \mathsf{f}(\mathsf{b}) &\approx \mathsf{g}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$

Example

$$\begin{aligned} f(g(f(x))) &\approx g(g(x)) \\ f(b) &\approx g(f(a)) \end{aligned}$$

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- simplify: $f(g(f(x))) \to g(f(f(x)))$

Example

$$\mathsf{g}(\mathsf{f}(\mathsf{f}(x))) \approx \mathsf{g}(\mathsf{g}(x))$$
$$\mathsf{f}(\mathsf{b}) \approx \mathsf{g}(\mathsf{f}(\mathsf{a}))$$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b}$$
$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{f}(x))$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- simplify: $\mathsf{g}(\mathsf{f}(\mathsf{f}(x))) \to \mathsf{g}(\mathsf{g}(x))$

Example

$$\begin{aligned} g(g(x)) &\approx g(g(x)) \\ f(b) &\approx g(f(a)) \end{aligned}$$

$$\begin{aligned} f(f(x)) &\rightarrow g(x) \\ g(a) &\rightarrow b \\ f(g(x)) &\rightarrow g(f(x)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- delete: $g(g(x)) = g(g(x))$

Example

$$f(b) \approx g(f(a))$$

$$\begin{aligned} f(f(x)) &\rightarrow g(x) \\ g(a) &\rightarrow b \\ f(g(x)) &\rightarrow g(f(x)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- orient: $f(b) >_{lpo} f(g(a))$

Example

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \\ \mathsf{f}(\mathsf{b}) &\to \mathsf{g}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- orient: $\mathsf{f}(\mathsf{b}) >_{\mathsf{lpo}} \mathsf{f}(\mathsf{g}(\mathsf{a}))$

Example

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \\ \mathsf{f}(\mathsf{b}) &\to \mathsf{g}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce: $\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a}))) \leftarrow \mathsf{f}(\mathsf{f}(\mathsf{b})) \to \mathsf{g}(\mathsf{b})$ critical pair

Example

$$\mathsf{f(g(f(a)))} \approx \mathsf{g(b)}$$

$$\begin{aligned} \mathsf{f(f(}x\mathsf{))} &\to \mathsf{g(}x\mathsf{)} \\ \mathsf{g(a)} &\to \mathsf{b} \\ \mathsf{f(g(}x\mathsf{))} &\to \mathsf{g(f(}x\mathsf{))} \\ \mathsf{f(b)} &\to \mathsf{g(f(a))} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce: $\mathsf{f(g(f(a)))} \leftarrow \mathsf{f(f(b))} \rightarrow \mathsf{g(b)}$ critical pair

Example

$$f(g(f(a))) \approx g(b)$$

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \\ f(b) &\to g(f(a)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- simplify: $f(g(f(a))) \to g(f(f(a)))$

Example

$$\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{g}(\mathsf{b})$$

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \\ \mathsf{f}(\mathsf{b}) &\to \mathsf{g}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- simplify: $\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \to \mathsf{g}(\mathsf{g}(\mathsf{a}))$

Example

$$g(g(a)) \approx g(b)$$

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \\ f(b) &\to g(f(a)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- simplify: $g(g(a)) \to g(b)$

Example

$$g(b) \approx g(b)$$

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \\ f(b) &\to g(f(a)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- delete: $g(b) = g(b)$

Example

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \\ f(b) &\to g(f(a)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- complete and reduced TRS

Example

$$\begin{aligned} f(f(x)) &\approx g(x) \\ g(a) &\approx b \end{aligned}$$

- LPO with precedence $f > g > b > a$

Example

$$\begin{aligned} f(f(x)) &\approx g(x) \\ g(a) &\approx b \end{aligned}$$

- LPO with precedence $b > g > f > a$

Example

$$\mathsf{f}(\mathsf{f}(x)) \approx \mathsf{g}(x)$$
$$\mathsf{g}(\mathsf{a}) \approx \mathsf{b}$$

- LPO with precedence $\mathsf{b} > \mathsf{g} > \mathsf{f} > \mathsf{a}$
- orient: $\mathsf{g}(x) >_{\mathsf{lpo}} \mathsf{f}(\mathsf{f}(x))$

Example

$$\mathsf{g}(x) \;\to\; \mathsf{f}(\mathsf{f}(x))$$

$$\mathsf{g}(\mathsf{a}) \;\approx\; \mathsf{b}$$

- LPO with precedence $\mathsf{b} > \mathsf{g} > \mathsf{f} > \mathsf{a}$
- orient: $\mathsf{g}(x) >_{\mathsf{lpo}} \mathsf{f}(\mathsf{f}(x))$

Example

$$g(x) \rightarrow f(f(x))$$

$$g(a) \approx b$$

- LPO with precedence $b > g > f > a$
- orient: $b >_{lpo} g(a)$

Example

$$\begin{aligned} g(x) &\rightarrow f(f(x)) \\ b &\rightarrow g(a) \end{aligned}$$

- LPO with precedence $b > g > f > a$
- orient: $b >_{lpo} g(a)$

Example

$$\begin{aligned} g(x) &\to f(f(x)) \\ b &\to g(a) \end{aligned}$$

- LPO with precedence $b > g > f > a$
- complete TRS

Example

$$\begin{aligned} g(x) &\to f(f(x)) \\ b &\to g(a) \end{aligned}$$

- LPO with precedence $b > g > f > a$
- complete TRS but not reduced

Example

$$\begin{aligned} g(x) &\to f(f(x)) \\ b &\to g(a) \end{aligned}$$

- LPO with precedence $b > g > f > a$
- compose: $g(a) \to f(f(a))$

Example

$$\begin{aligned} g(x) &\to f(f(x)) \\ b &\to f(f(a)) \end{aligned}$$

- LPO with precedence $b > g > f > a$
- compose: $g(a) \to f(f(a))$

Example

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- LPO with precedence $b > g > f > a$
- complete and reduced TRS

Example

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- orient: $g(x) >_{lpo} f(f(x))$

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Theorem

if complete reduced TRSs $\mathcal{R}$ *and* $\mathcal{S}$ *satisfy*

1. $\xleftrightarrow[\mathcal{R}]{*} \; = \; \xleftrightarrow[\mathcal{S}]{*}$
2. $\mathcal{R}$ *and* $\mathcal{S}$ *are compatible with same reduction order*

then $\mathcal{R} = \mathcal{S}$ *(modulo variable renaming)*

Outline

- Efficient Completion
- Cola Gene Puzzle
- Abstract Completion
- Proof Orders
- Critical Pair Criteria
- Further Reading

A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

TAGCTAGCTAGCT

in every fertilized egg into the cola gene

CTGACTGACT

Techniques exist to perform the following DNA substitutions

TCAT ↔ T   GAG ↔ AG   CTC ↔ TC   AGTA ↔ A   TAT ↔ CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

CTGCTACTGACT

What now, if unintendedly cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

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Example (Cola Gene Puzzle)

ES $\mathcal{E}$

$$\text{TCAT} \approx \text{T} \qquad \text{GAG} \approx \text{AG} \qquad \text{CTC} \approx \text{TC} \qquad \text{AGTA} \approx \text{A} \qquad \text{TAT} \approx \text{CT}$$

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- (milk gene) $\mathsf{TAGCTAGCTAGCT} \xleftrightarrow[\mathcal{E}]{*} \mathsf{CTGACTGACT}$ (cola gene)

$$\mathsf{TAGCTAGCTAGCT} \xrightarrow[\mathcal{R}]{!} \mathsf{T} \xleftarrow[\mathcal{R}]{!} \mathsf{CTGACTGACT}$$

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$$\mathsf{TAGCTAGCTAGCT} \xrightarrow[\mathcal{R}]{!} \mathsf{T} \neq \mathsf{TGT} \xleftarrow[\mathcal{R}]{!} \mathsf{CTGCTACTGACT}$$

Outline

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Definition

inference system $\mathcal{SC}$ (standard completion) consists of six rules

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Definitions

- $\trianglerighteq$ encompassment

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Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \triangleright \mathsf{s}(x) + y$

Definitions

- $\trianglerighteq$ encompassment

  $s \trianglerighteq t \iff \exists$ position p $\exists$ substitution σ: $s|_p = t\sigma$

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$\mathsf{s}(x) + \mathsf{s}(y + 0) \vartriangleright \mathsf{s}(x) + y \qquad x + x \vartriangleright x + y$

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Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \vartriangleright \mathsf{s}(x) + y \qquad x + x \vartriangleright x + y \qquad x + y \not\vartriangleright x + x$

Definition

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

inference system $\mathcal{SC}$ (standard completion) consists of six rules

delete $$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

compose $$\frac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$$ if $t \to_{\mathcal{R}} u$

simplify $$\frac{\mathcal{E} \cup \{s \dot{\approx} t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$$ if $t \to_{\mathcal{R}} u$

orient $$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$$ if $s > t$

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deduce $$\frac{\mathcal{E}, \mathcal{R}}{}$$ if $s \leftarrow_{\mathcal{R}} u \to_{\mathcal{R}} t$

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Definition

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

inference system $\mathcal{BC}$ (basic completion) consists of four rules

delete $$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

simplify $$\frac{\mathcal{E} \cup \{s \dot{\approx} t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$$ if $t \rightarrow_{\mathcal{R}} u$

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Definitions

- completion procedure is program that takes as input set of equations $\mathcal{E}$ and reduction order $>$ and generates (finite or infinite) sequence

$$(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$$

  with $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \varnothing$

Definitions

- completion procedure is program that takes as input set of equations $\mathcal{E}$ and reduction order $>$ and generates (finite or infinite) run

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  - run succeeds if $\mathcal{E}_\omega = \varnothing$ and $\mathcal{R}_\omega$ is confluent and terminating

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  - run succeeds if $\mathcal{E}_\omega = \varnothing$ and $\mathcal{R}_\omega$ is confluent and terminating
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- completion procedure is correct if every run that does not fail succeeds

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Question

how to guarantee correctness ?

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

Lemmata

- *if* $(\mathcal{E}, \mathcal{R}) \vdash_{\mathcal{SC}} (\mathcal{E}', \mathcal{R}')$ *and* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}' \subseteq >$

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- *if* $(\mathcal{E}, \mathcal{R}) \vdash_{\mathcal{SC}} (\mathcal{E}', \mathcal{R}')$ *then* $\xleftrightarrow[\mathcal{E} \cup \mathcal{R}]{*} = \xleftrightarrow[\mathcal{E}' \cup \mathcal{R}']{*}$

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

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Definition

$$\mathcal{E}_\infty = \bigcup_{i \geqslant 0} \mathcal{E}_i \quad \text{and} \quad \mathcal{R}_\infty = \bigcup_{i \geqslant 0} \mathcal{R}_i$$

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Lemmata

- $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty$

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

Lemmata

- *if* $(\mathcal{E}, \mathcal{R}) \vdash_{\mathcal{SC}} (\mathcal{E}', \mathcal{R}')$ *and* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}' \subseteq >$
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- $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty \subseteq >$
- $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[\mathcal{E}_\infty \cup \mathcal{R}_\infty]{*}$

Two Questions

non-failing run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

1. is $\mathcal{R}_\omega$ confluent ?

Two Questions

non-failing run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

1. is $\mathcal{R}_\omega$ confluent ?
2. $\xleftrightarrow[\mathcal{E}_\infty \cup \mathcal{R}_\infty]{*} = \xleftrightarrow[\mathcal{R}_\omega]{*}$?

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1. is $\mathcal{R}_\omega$ confluent ?
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Definitions

- run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$ is fair if

$$\mathrm{CP}(\mathcal{R}_\omega) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$$

Two Questions

non-failing run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

1. is $\mathcal{R}_\omega$ confluent ?
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- run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$ is fair if

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- completion procedure is fair if every run that does not fail is fair

Two Questions

non-failing run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

1. is $\mathcal{R}_\omega$ confluent ?
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Definitions

- run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$ is fair if

$$\mathsf{CP}(\mathcal{R}_\omega) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$$

- completion procedure is fair if every run that does not fail is fair

Theorem

every fair completion procedure is correct

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

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Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$

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Example

▸ skip

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$$\mathsf{a} \approx \mathsf{c}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- deduce: $\mathsf{a} \leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}) \to \mathsf{c}$

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▸ skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\approx \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\approx \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- deduce: $\mathsf{a} \leftarrow \mathsf{f}(\mathsf{c}, \mathsf{g}(y)) \to \mathsf{f}(\mathsf{g}(\mathsf{c}), y)$

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Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\approx \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{a} >_{\mathsf{lpo}} \mathsf{c}$

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Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

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collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x,\mathsf{c}) &\to x \\ \mathsf{f}(x,\mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x),y) \\ \mathsf{f}(\mathsf{c},y) &\to \mathsf{a} \end{aligned} \qquad \begin{aligned} \mathsf{a} &\to \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}),y) &\to \mathsf{a} \\ \mathsf{a} &\approx \mathsf{g}(\mathsf{c}) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- deduce: $\mathsf{a} \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{c}) \to \mathsf{g}(\mathsf{c})$

Remark

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collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\to \mathsf{a} \\ \mathsf{a} &\approx \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\approx \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- deduce: $\mathsf{a} \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{c}), \mathsf{g}(y)) \to \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\to \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\approx \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{a} >_{\mathsf{lpo}} \mathsf{g}(\mathsf{c})$

Remark

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collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned} \qquad \begin{aligned} \mathsf{a} &\to \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\to \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) >_{\mathsf{lpo}} \mathsf{a}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\quad \dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}} \quad$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned} \qquad \begin{aligned} \mathsf{g}(\mathsf{c}) &\approx \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\to \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- collapse: $\quad \mathsf{a} \to \mathsf{g}(\mathsf{c})$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{c} &\approx \mathsf{c} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\to \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- simplify: $\mathsf{g}(\mathsf{c}) \to \mathsf{c}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x,\mathsf{c}) &\to x \\ \mathsf{f}(x,\mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x),y) \\ \mathsf{f}(\mathsf{c},y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{f}(\mathsf{g}(\mathsf{c}),y) &\to \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})),y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- delete

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{f}(\mathsf{c}, y) &\approx \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- collapse: $\mathsf{f}(\mathsf{g}(\mathsf{c}), y) \to \mathsf{f}(\mathsf{c}, y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\approx \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- simplify: $\mathsf{f}(\mathsf{c}, y) \to \mathsf{a}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- delete

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$
\begin{aligned}
\mathsf{a} &\to \mathsf{b} \\
\mathsf{g}(x) &\to x \\
\mathsf{f}(x, \mathsf{c}) &\to x \\
\mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\
\mathsf{f}(\mathsf{c}, y) &\to \mathsf{a}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{a} &\approx \mathsf{g}(\mathsf{g}(\mathsf{c})) \\
\mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\
\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a}
\end{aligned}
$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- deduce: $\mathsf{a} \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), \mathsf{c}) \to \mathsf{g}(\mathsf{g}(\mathsf{c}))$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x,\mathsf{c}) &\to x \\ \mathsf{f}(x,\mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x),y) \\ \mathsf{f}(\mathsf{c},y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\approx \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))),y) &\approx \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})),y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- deduce: $\mathsf{a} \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})),\mathsf{g}(y)) \to \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))),y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x,\mathsf{c}) &\to x \\ \mathsf{f}(x,\mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x),y) \\ \mathsf{f}(\mathsf{c},y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))),y) &\approx \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})),y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{a} >_{\mathsf{lpo}} \mathsf{g}(\mathsf{g}(\mathsf{c}))$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \\ \mathsf{a} &\to \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) >_{\mathsf{lpo}} \mathsf{a}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \\ \mathsf{g}(\mathsf{g}(\mathsf{c})) &\approx \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- collapse: $\mathsf{a} \to \mathsf{g}(\mathsf{g}(\mathsf{c}))$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned} \qquad \begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \\ \mathsf{g}(\mathsf{c}) &\approx \mathsf{g}(\mathsf{c}) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

skip

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- simplify: $\mathsf{g}(\mathsf{g}(\mathsf{c})) \to \mathsf{g}(\mathsf{c})$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \\ \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- delete

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \\ \mathsf{f}(\mathsf{g}(\mathsf{c}), y) &\approx \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- collapse: $\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{c})), y) \to \mathsf{f}(\mathsf{g}(\mathsf{c}), y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \\ \\ \mathsf{f}(\mathsf{c}, y) &\approx \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- simplify: $\mathsf{f}(\mathsf{g}(\mathsf{c}), y) \to \mathsf{f}(\mathsf{c}, y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \\ \\ \mathsf{a} &\approx \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- simplify: $\mathsf{f}(\mathsf{c}, y) \to \mathsf{a}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x,\mathsf{c}) &\to x \\ \mathsf{f}(x,\mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x),y) \\ \mathsf{f}(\mathsf{c},y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))),y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- delete

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- $\dots$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

skip

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

$$\begin{aligned} \mathsf{a} &\to \mathsf{g}(\mathsf{g}(\mathsf{c})) \\ \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{g}(\mathsf{c}))), y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- $\cdots$ fair but unsuccessful run

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{a} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(\mathsf{g}(x), y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- compose: $\mathsf{a} \to \mathsf{b}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, \mathsf{g}(y)) &\to \mathsf{f}(x, y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- compose: $\mathsf{f}(\mathsf{g}(x), y) \to \mathsf{f}(x, y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \mathsf{f}(x, y) &\approx \mathsf{f}(x, y) \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- collapse: $\mathsf{f}(x, \mathsf{g}(y)) \to \mathsf{f}(x, y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- delete

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned}$$

$$\mathsf{c} \approx \mathsf{b}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- deduce: $\mathsf{c} \leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}) \to \mathsf{b}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{c}) &\to x \\ \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned}$$

$$\mathsf{c} \to \mathsf{b}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{c} >_{\mathsf{lpo}} \mathsf{b}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{b}) &\approx x \\ \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned}$$

$\mathsf{c} \to \mathsf{b}$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- collapse: $\mathsf{f}(x, \mathsf{c}) \to \mathsf{f}(x, \mathsf{b})$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{b}) &\to x \\ \\ \mathsf{f}(\mathsf{c}, y) &\to \mathsf{b} \end{aligned} \qquad \mathsf{c} \to \mathsf{b}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{f}(x, \mathsf{b}) >_{\mathsf{lpo}} x$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{b}) &\to x \\ \\ \mathsf{f}(\mathsf{b}, y) &\approx \mathsf{b} \end{aligned}$$

$$\mathsf{c} \to \mathsf{b}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- collapse: $\mathsf{f}(\mathsf{c}, y) \to \mathsf{f}(\mathsf{b}, y)$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{b}) &\to x \\ \\ \mathsf{f}(\mathsf{b}, y) &\to \mathsf{b} \end{aligned}$$

$$\mathsf{c} \to \mathsf{b}$$

- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- orient: $\mathsf{f}(\mathsf{b}, y) >_{\mathsf{lpo}} \mathsf{b}$

Remark

strict encompassment condition in collapse rule cannot be dropped

collapse $\quad \dfrac{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}} \quad$ if $t \to_{\mathcal{R}} u$ using $\ell \to r \in \mathcal{R}$ with $t \rhd \ell$

Example

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{g}(x) &\to x \\ \mathsf{f}(x, \mathsf{b}) &\to x \\ \\ \mathsf{f}(\mathsf{b}, y) &\to \mathsf{b} \end{aligned} \qquad \mathsf{c} \to \mathsf{b}$$

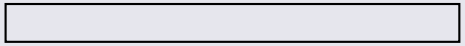
- LPO with precedence $\mathsf{f} > \mathsf{a} > \mathsf{g} > \mathsf{c} > \mathsf{b}$
- complete and reduced TRS

Outline

- Efficient Completion
- Cola Gene Puzzle
- Abstract Completion
- **Proof Orders**
- Critical Pair Criteria
- Further Reading

Completion is Proof Normalization

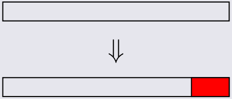
$(\mathcal{E}, \mathcal{R})$



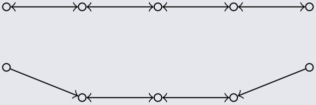
proof in $(\mathcal{E}, \mathcal{R})$

Completion is Proof Normalization

$(\mathcal{E}, \mathcal{R})$

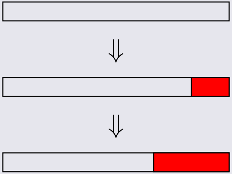


proof in $(\mathcal{E}, \mathcal{R})$

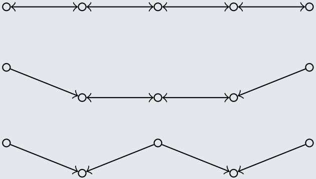


Completion is Proof Normalization

$(\mathcal{E}, \mathcal{R})$

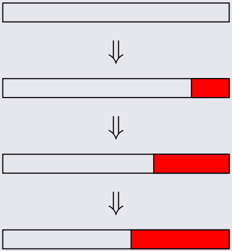


proof in $(\mathcal{E}, \mathcal{R})$

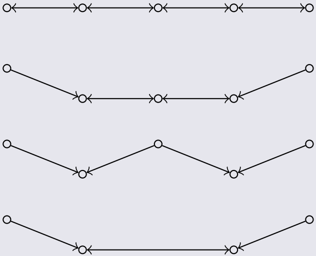


Completion is Proof Normalization

$(\mathcal{E}, \mathcal{R})$

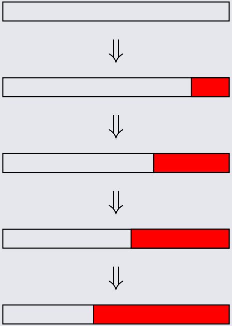


proof in $(\mathcal{E}, \mathcal{R})$

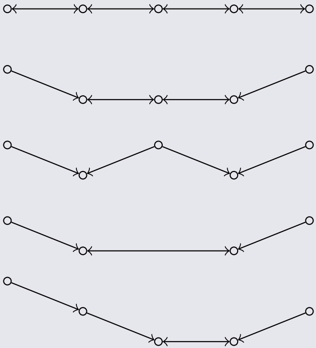


Completion is Proof Normalization

$(\mathcal{E}, \mathcal{R})$

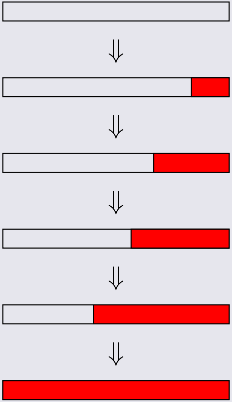


proof in $(\mathcal{E}, \mathcal{R})$



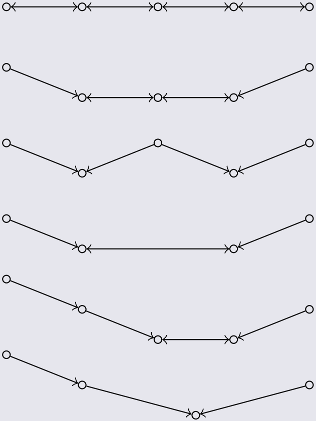
Completion is Proof Normalization

$(\mathcal{E}, \mathcal{R})$



fair derivation

proof in $(\mathcal{E}, \mathcal{R})$



rewrite proof

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

Definitions

- proof of $s \approx t$ is sequence $(u_1, \ldots, u_n)$ of terms such that
  - $s = u_1$
  - $t = u_n$
  - for all $1 \leqslant i < n$ $u_i \rightarrow_{\mathcal{R}} u_{i+1}$ or $u_i \leftarrow_{\mathcal{R}} u_{i+1}$ or $u_i \leftrightarrow_{\mathcal{E}} u_{i+1}$

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  for some $1 \leqslant j \leqslant n$
- two proofs $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ are equivalent if $s_1 = t_1$ and $s_n = t_n$

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Lemma

$\gg_{mul}$ is a well-founded order on proofs

non-failing and fair run $\mathcal{S}$: $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

Lemma

$\forall$ *proof* P *in* $\mathcal{E}_\infty \cup \mathcal{R}_\infty$ *that is no rewrite proof in* $\mathcal{R}_\omega$
$\exists$ *equivalent proof* Q *in* $\mathcal{E}_\infty \cup \mathcal{R}_\infty$ *such that* $P \gg_{mul} Q$

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Proof Sketch

three cases:

1. P contains step using equation $\ell \approx r \in \mathcal{E}_\infty$

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   use critical pair lemma

Theorem

$\forall$ *non-failing and fair run* $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$

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Corollary

every fair completion procedure is correct

Outline

- Efficient Completion
- Cola Gene Puzzle
- Abstract Completion
- Proof Orders
- **Critical Pair Criteria**
- Further Reading

Fact

$\mathrm{CP}(\mathcal{R}_\omega) \subseteq \mathcal{E}_\infty$ ensures correcteness

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- critical pair criterion CPC is correct if $\mathcal{R}_\omega$ is confluent and terminating for every non-failing run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathcal{SC}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathcal{SC}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathcal{SC}} \cdots$ that is fair with respect to critical pair criterion CPC

Definitions

- peak $P\colon s \leftarrow_{\mathcal{R}} u \rightarrow_{\mathcal{R}} t$ is composite if there exist proofs

$$Q_1\colon u_1 \xleftrightarrow{*} u_2 \quad \cdots \quad Q_{n-1}\colon u_{n-1} \xleftrightarrow{*} u_n$$

  such that

  - $s = u_1$
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Definition

composite critical pair criterion: $\mathrm{CCP}(\mathcal{E}) = \{s \approx t \in \mathrm{CP}(\mathcal{E}) \mid s \approx t \text{ is composite}\}$

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critical pair criterion CCP is correct

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- critical pair $s \leftarrow\rtimes\rightarrow t$ originating from overlap $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ with mgu σ is unblocked if $x\sigma$ is reducible for some $x \in \mathcal{V}\mathrm{ar}(\ell_1) \cup \mathcal{V}\mathrm{ar}(\ell_2)$

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Lemma

- *every unblocked critical pair is composite*

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- critical pair $s \leftarrow\!\rtimes\!\rightarrow t$ originating from overlap $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ with mgu σ is reducible if proper subterm of $\ell_1\sigma$ is reducible

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Lemma

- *every unblocked critical pair is composite*
- *every reducible critical pair is composite*

Example

TRS

$$\begin{aligned} e^- &\to e & x/e &\to x \\ x^{--} &\to x & e/x &\to x \\ x \cdot (x^- \cdot y) &\to y & (x/y^-)/y &\to x \\ x^- &\to e/x & z/(z^-/y)^- &\to y^- \end{aligned}$$

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$$e^- \to e \qquad x/e \to x$$
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critical pair

$$y/e^- \leftarrow \rtimes \rightarrow y$$

originating from overlap

$$\langle\, x/e \to x,\ \epsilon,\ (y/z^-)/z \to y \,\rangle$$

Example

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$$\mathsf{e}^- \to \mathsf{e} \qquad x/\mathsf{e} \to x$$
$$x^{--} \to x \qquad \mathsf{e}/x \to x$$
$$x \cdot (x^- \cdot y) \to y \qquad (x/y^-)/y \to x$$
$$x^- \to \mathsf{e}/x \qquad z/(z^-/y)^- \to y^-$$

critical pair

$$y/\mathsf{e}^- \leftarrow \rtimes \to y$$

originating from overlap

$$\langle\, x/\mathsf{e} \to x,\ \epsilon,\ (y/z^-)/z \to y \,\rangle$$

is reducible because $(y/\mathsf{e}^-)/\mathsf{e}$ is reducible at position 12

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Canonical Equational Proofs
Leo Bachmair
Progress in Theoretical Computer Science, Birkhäuser, 1991

Equational Inference, Canonical Proofs, and Proof Orderings
Leo Bachmair and Nachum Dershowitz
J.ACM 41(2), pp. 236–276, 1994

Completion Tools

- Waldmeister
- Slothrop
- mkbTT
- KBCV