



Introduction to Term Rewriting

lecture 7

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, **termination**

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Knuth-Bendix Order
- Dependency Pairs
- Further Reading

Definition

rewrite system is **terminating** if there are no infinite rewrite sequences

Termination Methods

Knuth-Bendix order, **polynomial interpretations**, multiset order, simple path order, **lexicographic path order**, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, **well-founded monotone algebras**, general path order, semantic labeling, dummy elimination, **dependency pairs**, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, ...

Outline

- Knuth-Bendix Order
- Dependency Pairs
- Further Reading

Definitions

- **weight function** (w, w_0) consists of mapping $w: \mathcal{F} \rightarrow \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geq w_0$ for all constants $c \in \mathcal{F}$
- **weight** of term t is

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \sum_{i=1}^n w(t_i) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

- weight function (w, w_0) is **admissible** for precedence $>$ if

$$f > g \quad \forall g \in \mathcal{F} \setminus \{f\}$$

whenever f is unary function symbol in \mathcal{F} with $w(f) = 0$

Example

- rewrite rules

$$\begin{array}{ll}
 e \cdot x \rightarrow x & x \cdot e \rightarrow x \\
 x^- \cdot x \rightarrow e & x \cdot x^- \rightarrow e \\
 (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) & x^{--} \rightarrow x \\
 e^- \rightarrow e & (x \cdot y)^- \rightarrow y^- \cdot x^- \\
 x^- \cdot (x \cdot y) \rightarrow y & x \cdot (x^- \cdot y) \rightarrow y
 \end{array}$$

- weight function $w(e) = w(\cdot) = w_0 = 1$ $w(^-) = 0$

$$w(e \cdot x) = 3 \quad w(x) = 1 \quad w((x \cdot y)^-) = 3$$

- precedence $^- > \cdot > e$
- admissible because $^-$ is maximal in precedence

Definition

- relation $>_{\text{kbo}}$ (**Knuth-Bendix order**) on terms:

$s >_{\text{kbo}} t$ if $|s|_x \geq |t|_x$ for all $x \in \mathcal{V}$ and either

① $w(s) > w(t)$

② $w(s) = w(t)$ and either

① $\exists n > 0$ such that $s = f^n(t)$ and $t \in \mathcal{V}$

② $s = f(s_1, \dots, s_n)$ and $t = f(t_1, \dots, t_n)$ and $\exists i$

$$\forall j < i \quad s_j = t_j \quad s_i >_{\text{kbo}} t_i$$

③ $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f > g$

Theorem

$>_{\text{kbo}}$ is **reduction order** if precedence $>$ is well-founded

Example (1)

- rewrite rules

$$\begin{array}{ll}
 e \cdot x \rightarrow x & x \cdot e \rightarrow x \\
 x^- \cdot x \rightarrow e & x \cdot x^- \rightarrow e \\
 (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) & x^{--} \rightarrow x \\
 e^- \rightarrow e & (x \cdot y)^- \rightarrow y^- \cdot x^- \\
 x^- \cdot (x \cdot y) \rightarrow y & x \cdot (x^- \cdot y) \rightarrow y
 \end{array}$$

- weight function $w(e) = w(\cdot) = w_0 = 1$ $w(^-) = 0$
- precedence $^- > \cdot > e$

$$e \cdot x >_{\text{kbo}} x \quad x^{--} >_{\text{kbo}} x \quad (x \cdot y)^- >_{\text{kbo}} y^- \cdot x^-$$

Example (2)

- rewrite rules

$$aa \rightarrow bbb \quad bbbbb \rightarrow aaa$$

- weight function and precedence

$$w(a) = 3 \quad w(b) = 2 \quad a > b$$

$$w(a) = 5 \quad w(b) = 3 \quad b > a$$

$$w(a) = 13 \quad w(b) = 8$$

Example (3)

- rewrite rules

$$\begin{array}{llll}
 0 + 0 \rightarrow 0 & 1 + 0 \rightarrow 1 & \dots & 9 + 0 \rightarrow 9 \\
 0 + 1 \rightarrow 1 & 1 + 1 \rightarrow 2 & \dots & 9 + 1 \rightarrow 1 : 0 \\
 & \vdots & & \vdots \\
 0 + 9 \rightarrow 9 & 1 + 9 \rightarrow 1 : 0 & \dots & 9 + 9 \rightarrow 1 : 8 \\
 x + (y : z) \rightarrow y : (x + z) & & & 0 : x \rightarrow x \\
 (x : y) + z \rightarrow x : (y + z) & & & x : (y : z) \rightarrow (x + y) : z
 \end{array}$$

- weight function

$$\begin{array}{l}
 w(0) = w(1) = w(2) = w(3) = w(4) = w(+) = w_0 = 1 \\
 w(:) = 2 \quad w(5) = w(6) = w(7) = w(8) = w(9) = 3
 \end{array}$$

- precedence $+ > :$ $+ > 5$ $+ > 6$ $+ > 7$ $+ > 8$

Example (4)

- rewrite rules

$$11 \rightarrow 43 \qquad 33 \rightarrow 56 \qquad 55 \rightarrow 62$$

$$12 \rightarrow 21 \qquad 34 \rightarrow 11 \qquad 56 \rightarrow 12$$

$$22 \rightarrow 111 \qquad 44 \rightarrow 3 \qquad 66 \rightarrow 21$$

- weight function and precedence

$$w(1) = 31$$

$$w(2) = 47$$

$$w(3) = 41$$

$$w(4) = 21$$

$$w(5) = 43$$

$$w(6) = 3$$

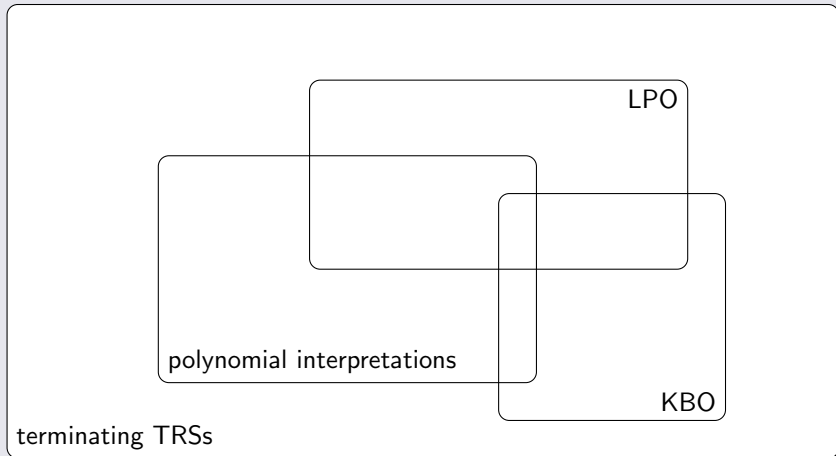
$$3 > 5 > 6 > 1 > 4 \qquad 1 > 2$$

Theorem

- if $> \subseteq \sqsupset$ then $>_{kbo} \subseteq \sqsupset_{kbo}$ (*incrementality*)
- if $>$ is total then $>_{kbo}$ is *total on ground terms*
- following two problems are *decidable*:
 - 1 instance: terms s, t weight function (w, w_0) precedence $>$
question: $s >_{kbo} t ?$
 - 2 instance: terms s, t
question: \exists weight function (w, w_0) such that $s >_{kbo} t ?$
 \exists precedence $>$

Remark

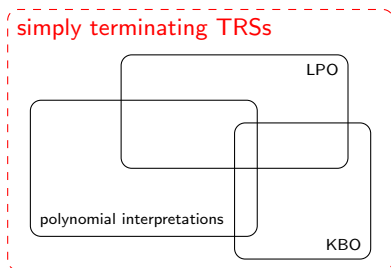
KBO, LPO and polynomial interpretations are incomparable



Outline

- Knuth-Bendix Order
- **Dependency Pairs**
- Further Reading

More Realistic View



terminating TRSs

dependency pairs make direct termination methods much more powerful

Definitions

TRS \mathcal{R} over signature \mathcal{F}

- $\mathcal{F}^\# = \mathcal{F} \cup \{f^\# \mid f \text{ is defined symbol of } \mathcal{R}\}$
- if $t = f(t_1, \dots, t_n)$ with f defined then $t^\# = f^\#(t_1, \dots, t_n)$
- **dependency pair** $l^\# \rightarrow u^\#$ of rewrite rule $l \rightarrow r$ satisfies
 - $u \sqsubseteq r$ and $u \not\prec l$
 - $\text{root}(u)$ is defined symbol
- **DP(\mathcal{R})** is set of all dependency pairs of \mathcal{R}

Theorem

\forall *non-terminating TRS* \mathcal{R} \exists *infinite rewrite sequence*

$$t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[DP(\mathcal{R})]{\epsilon} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[DP(\mathcal{R})]{\epsilon} \dots$$

such that t_1 is terminating with respect to \mathcal{R}

Example

rewrite rule

$$f(g(x)) \rightarrow g(g(f(f(x)))) \quad \mathcal{R}$$

infinite rewrite sequence in \mathcal{R}

$$f(g(g(x))) \xrightarrow{\mathcal{R}} g(g(f(f(g(x)))))) \xrightarrow{\mathcal{R}} g(g(f(g(g(f(f(x))))))) \xrightarrow{\mathcal{R}} \dots$$

dependency pairs

$$\begin{aligned} f^\#(g(x)) &\rightarrow f^\#(f(x)) \\ f^\#(g(x)) &\rightarrow f^\#(x) \end{aligned} \quad \text{DP}(\mathcal{R})$$

infinite sequence in $\mathcal{R} \cup \text{DP}(\mathcal{R})$

$$f^\#(g(g(x))) \xrightarrow{\text{DP}(\mathcal{R})} f^\#(f(g(x))) \xrightarrow{\mathcal{R}} f^\#(g(g(f(f(x)))))) \xrightarrow{\text{DP}(\mathcal{R})} \dots$$

Example

rewrite rules

$$\begin{array}{ll}
 0 + y \rightarrow y & 0 \times y \rightarrow 0 \\
 s(x) + y \rightarrow s(x + y) & s(x) \times y \rightarrow (x \times y) + y
 \end{array}$$

dependency pairs

$$\begin{array}{l}
 s(x) +^{\#} y \rightarrow x +^{\#} y \\
 s(x) \times^{\#} y \rightarrow (x \times y) +^{\#} y \\
 s(x) \times^{\#} y \rightarrow x \times^{\#} y
 \end{array}$$

polynomial interpretation

$$\begin{array}{l}
 0_{\mathbb{N}} = 0 \\
 s_{\mathbb{N}}(x) = x + 1 \\
 +_{\mathbb{N}}(x, y) = x + y \\
 \times_{\mathbb{N}}(x, y) = +_{\mathbb{N}}^{\#}(x, y) = \times_{\mathbb{N}}^{\#}(x, y) = x
 \end{array}$$

Theorem

\forall non-terminating TRS \mathcal{R} \exists infinite rewrite sequence

$$t_1 \succsim t_2 > t_3 \succsim t_4 > \dots$$

such that t_1 is terminating with respect to \mathcal{R}

Definition

reduction pair $(>, \succsim)$ consists of well-founded order $>$ and preorder \succsim such that

- 1 $>$ is closed under substitutions
- 2 \succsim is closed under contexts and substitutions
- 3 $> \cdot \succsim \subseteq >$ or $\succsim \cdot > \subseteq >$

Theorem

TRS \mathcal{R} is terminating $\iff DP(\mathcal{R}) \subseteq >$ and $\mathcal{R} \subseteq \succsim$ for reduction pair $(>, \succsim)$

Example

rewrite rules

$$\begin{array}{ll}
 x - 0 \rightarrow 0 & 0 \div s(y) \rightarrow 0 \\
 s(x) - s(y) \rightarrow x - y & s(x) \div s(y) \rightarrow s((x - y) \div s(y))
 \end{array}$$

dependency pairs

$$\begin{array}{l}
 s(x) - \# s(y) \rightarrow x - \# y \\
 s(x) \div \# s(y) \rightarrow (x - y) \div \# s(y) \\
 s(x) \div \# s(y) \rightarrow x - \# y
 \end{array}$$

polynomial interpretation

$$\begin{array}{l}
 0_{\mathbb{N}} = 0 \\
 s_{\mathbb{N}}(x) = x + 1 \\
 -_{\mathbb{N}}(x, y) = -\#_{\mathbb{N}}(x, y) = \div_{\mathbb{N}}(x, y) = \div\#_{\mathbb{N}}(x, y) = x
 \end{array}$$

Definitions

- **well-founded weakly monotone \mathcal{F} -algebra** $(\mathcal{A}, >)$ consists of nonempty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is **weakly monotone** in all coordinates:

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) \geq f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

for all $a_1, \dots, a_n, b \in A$ and $i \in \{1, \dots, n\}$ with $a_i > b$

- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α
- relation $\geq_{\mathcal{A}}$ on terms: $s \geq_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) \geq [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemma

$(>_{\mathcal{A}}, \geq_{\mathcal{A}})$ is reduction pair for every well-founded weakly monotone algebra $(\mathcal{A}, >)$

Example

rewrite rules

$$\text{average}(0, 0) \rightarrow 0$$

$$\text{average}(s(x), y) \rightarrow \text{average}(x, s(y))$$

$$\text{average}(0, s(0)) \rightarrow 0$$

$$\text{average}(x, s(s(s(y)))) \rightarrow s(\text{average}(s(x), y))$$

$$\text{average}(0, s(s(0))) \rightarrow s(0)$$

dependency pairs

$$\text{average}^\sharp(s(x), y) \rightarrow \text{average}^\sharp(x, s(y))$$

$$\text{average}^\sharp(x, s(s(s(y)))) \rightarrow \text{average}^\sharp(s(x), y)$$

polynomial interpretation

$$0_{\mathbb{N}} = 0$$

$$\text{average}_{\mathbb{N}}(x, y) = x + y$$

$$s_{\mathbb{N}}(x) = x + 1$$

$$\text{average}_{\mathbb{N}}^\sharp(x, y) = 2x + y$$

Outline

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- **Further Reading**



KBO Orientability

Harald Zankl, Nao Hirokawa and Aart Middeldorp
JAR 43(2), pp. 173 – 201, 2009



Termination of Term Rewriting using Dependency Pairs

Thomas Arts and Jürgen Giesl
TCS 236(1,2), pp. 133 – 178, 2000